Published in *Proceedings of the 47th IEEE Conference on Decision and Control*, Cancun, Mexico, 9-11 december 2008, which should be cited to refer to this work.
DOI: 10.1109/CDC.2008.4739344

Frequency Domain Controller Design by Linear Programming Guaranteeing Quadratic Stability

Marc Kunze, Alireza Karimi and Roland Longchamp

***Abstract*— In a recent work, a frequency method based on linear programming was proposed to design fixed-order linearly parameterized controllers for stable linear multi-model SISO systems. The method is based on the shaping of the open-loop transfer functions in the Nyquist diagram under a set of linear constraints guaranteeing a lower bound on the crossover frequency and a linear stability margin. In this paper, this method is extended to guarantee quadratic stability. For this purpose, new linear constraints based on the phase difference of the characteristic polynomials of the closed-loop systems are added in the Nyquist diagram. A simulation example illustrates the effectiveness of the proposed approach.**

I. Introduction

In a previous work [1], a frequency-domain controller design method based on linear programming was proposed. The open-loop transfer function is shaped in the Nyquist diagram using a set of linear constraints for robustness and for the lower bound of the crossover frequency. This method can directly compute controllers for multi-model systems assuring robust stability. As the loop-shaping of QFT controllers [2], this frequency-domain method does not ensure quadratic stability. It means that the closed-loop system is stable for anyone of the model belonging to the multi-model set as long as this model is fixed during operation. If the model varies during operation, the stability is not guaranteed anymore. An extension of this method is proposed in [3] to design gain-scheduled controller. Once again, the stability is guaranteed only for slow variation of models.

In the literature, a lot of methods can be found guaranteeing the quadratic stability of multi-model systems or the global stability of LPV systems (see the survey papers [4] and [5]). Most of these methods are based on a time-domain analysis using a quadratic Lyapunov function approach or a parameter-dependant quadratic Lyapunov function approach to reduce the conservativeness. Since these approaches are time-domain based, they are not compatible with our frequency-domain based method.

In this paper, we propose to use a theorem making the link between the small gain theorem and SPRness properties in order to transform conditions of quadratic stability in the time domain into conditions in the frequency domain. Then, these frequency conditions are transformed into linear constraints. Thus, it is possible to use the frequency method proposed in [1] and [3] to design controllers guaranteeing quadratic stability simply by adding linear constraints in the Nyquist diagram. For the moment, this method is restricted to SISO multi-model systems having two models in the set or to certain classes of SISO LPV systems. The extension of the method to multi-model systems having more than two models is the subject of future work.

This paper is organized as follows: in Section II the class of plant and controllers are defined and a summary of the frequency method based on linear programming is given. The theorem used to make the link between the time domain and frequency domain conditions and the linear constraints rising from it are developped in Section III. To show the effectiveness of the method, it is applied to a switched system in section IV. Finally, Section V gives some concluding remarks.

The authors are with the Laboratoire d'Automatique, Ecole Polytechnique Fédérale de Lausanne (EPFL), CH-1015 Lausanne, Switzerland
Corresponding author: alireza.karimi@epfl.ch

II. Preliminaries and Problem Formulation

A. Plant model

The class of stable SISO multi-model systems consisting of two continuous-time linear model is considered. Each model is defined by a transfer function of the form:

$$G_i(s) = \frac{G_{n,i}(s)}{G_{d,i}(s)} \quad i = 1, 2 \tag{1}$$

where i refers to the first or the second model. In this paper we consider continuous-time models, but it should be noted that the proposed approach can also be applied to discrete-time models straightforwardly.

B. Controller parameterization

The class of linearly parameterized controllers is considered:

$$K(s) = \frac{K_n(s)}{K_d(s)} = \rho^T \phi(s) \tag{2}$$

where

$$\rho^T = \begin{bmatrix} \rho_1 & \rho_2 & \dots & \rho_m \end{bmatrix} \tag{3}$$

$$\phi^T(s) = \begin{bmatrix} \phi_1(s) & \phi_2(s) & \dots & \phi_m(s) \end{bmatrix}. \tag{4}$$

m is the number of controller parameters and $\phi_l(s)$, $l = 1, \dots, m$, are transfer functions with no RHP pole. The main property of this parameterization is that every point on the Nyquist diagram of $L_i(j\omega) = K(j\omega)G_i(j\omega)$ can be written as a linear function of the controller parameters ρ:

$$\begin{aligned} K(j\omega_k)G_i(j\omega_k) &= \rho^T \phi(j\omega_k)G_i(j\omega_k) \\ &= \rho^T \mathcal{R}_i(\omega_k) + j\rho^T \mathcal{I}_i(\omega_k) \end{aligned} \tag{5}$$

where ω_k is a particular value of the frequency ω, $\mathcal{R}_i(\omega_k)$ and $\mathcal{I}_i(\omega_k)$ are, respectively, the real and imaginary parts of $\phi(j\omega_k)G_i(j\omega_k)$.

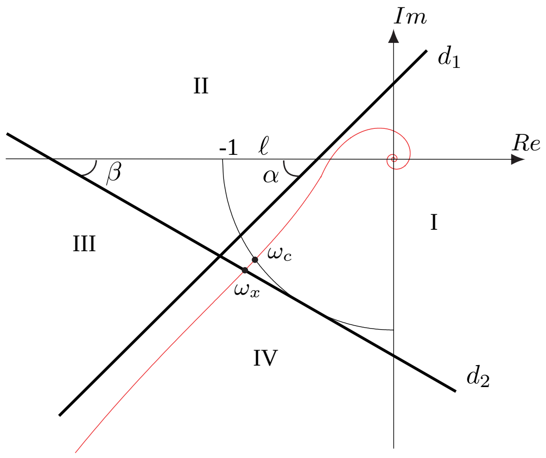

Fig. 1. Linear constraints for robustness and performance, with four regions: I, II, III and IV. The linear stability margin, the crossover frequency and the lower bound on the crossover frequency are, respectively, ℓ, ω_c and ω_x.

C. Summary of the Frequency Method Based on Linear Programming

Classical gain, phase and modulus margins as well as crossover frequency ω_c are nonlinear functions of the controller parameters. Optimization methods with constraints on these values lead to non-convex optimization problems. The frequency method based on linear programming introduces a new stability margin and a lower bound on the crossover frequency which lead to linear constraints for an optimization problem in which robustness or performance are maximized. This optimization problem can be solved efficiently by a linear programming method. The following is a summary and the reader can see [1] to have a full description of the method.

1) Linear robustness margin: Consider a straight line d_1 in the complex plane crossing the negative real axis between 0 and -1 with an angle $\alpha \in \ (0°, 90°]$ (see Fig. 1). The linear stability margin $\ell \in \ (0, 1)$ is the distance between the critical point -1 and d_1 where it crosses the negative real axis. If the Nyquist plot of the open-loop transfer function lies on the right-hand side of d_1, a lower bound on the conventional robustness margins is ensured.

2) Lower bound on the crossover frequency: Consider another straight line d_2 in the complex plane tangent to the unit circle centered at the origin which crosses the negative real axis with an angle β (see Fig. 1). The part of d_2 between d_1 and the imaginary axis is a linear approximation of the unit circle in this region. Now, assume that the open-loop Nyquist plot intersects d_2 at a frequency called ω_x. From Fig. 1 it is clear that the crossover frequency ω_c is always greater than or equal to ω_x. Hence, ω_x, a lower approximation of the crossover frequency, can be used as a measure of the time-domain performance or the closed-loop bandwidth.

3) Optimization for robustness and performance: Optimizing the robustness consists of maximizing the linear robustness margin ℓ.

Optimizing the load disturbance rejection is considered as the desired performance for the closed-loop system. In general, to reject low-frequency disturbances, the controller gain at low-frequencies should be maximized. For a rational continuous-time controller of order n_c with fixed denominator $K_d(s)$,

$$K(s) = \frac{k_{n_c}s^{n_c} + \ldots + k_1 s + k_0}{K_d(s)} \tag{6}$$

it corresponds to maximizing k_0. According to (2), k_0 is a linear combination of the parameters of the linearly parameterized controller:

$$k_0 = \sum_{l=1}^{m} \gamma_l \rho_l \tag{7}$$

where γ_l are the coefficients of the linear combination which depend on the basis functions.

For example, optimizing the load disturbance rejection of a multi-model system composed of two models with only one integrator in $L_i(j\omega)$ is expressed by:

$$\begin{aligned} &\text{maximize } k_0 \\ &\text{subject to:} \\ &\rho^T(\cot\alpha\,\mathcal{I}_i(\omega_k) - \mathcal{R}_i(\omega_k)) + \ell \leq 1 \quad \forall\omega_k,\ i = 1, 2. \end{aligned} \tag{8}$$

It corresponds to constraining $K(j\omega_k)G_i(j\omega_k)$ to be in region I or IV (see Fig. 1) for all frequencies.

III. Frequency Domain Conditions for Quadratic Stability

A. Quadratic stability of two systems

Let a stable polynomial of order n be defined as:

$$c(s) = s^n + c_1 s^{n-1} + \ldots + c_n. \tag{9}$$

We can assign to this polynomial a vector C:

$$C = \begin{bmatrix} c_1 & c_2 & \ldots & c_n \end{bmatrix} \tag{10}$$

and a matrix A:

$$A = \begin{bmatrix} -c_1 & -c_2 & \ldots & -c_{n-1} & -c_n \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix}. \tag{11}$$

Then the relation between strictly positive realness and the quadratic stability can be stated in the following theorem.

Theorem 1: Consider $c_1(s)$ and $c_2(s)$, two stable polynomials of order n, then the following are equivalent:

1) $\frac{c_1(s)}{c_2(s)}$ and $\frac{c_2(s)}{c_1(s)}$ are Strictly Positive Real (SPR).
2) $|\arg(c_1(j\omega)) - \arg(c_2(j\omega))| < \frac{\pi}{2} \quad \forall\omega.$
3) A_1 and A_2 are quadratically stable meaning that: $\exists P = P^T > 0 \in \mathbb{R}^{n\times n}$ such that

$$A_1^T P + PA_1 < 0 \quad , \quad A_2^T P + PA_2 < 0.$$

Proof: See Appendix A. ■

This theorem is useful to create new linear constraints in the Nyquist diagram to ensure the quadratic stability of multi-model systems.

B. Linear Constraints to Ensure Quadratic Stability

To ensure the quadratic stability of multi-model systems, the idea is to transform the properties given in Theorem 1 into linear constraints on the open-loop Nyquist curves. Let us consider the following ratio:

$$\frac{1+K(s)G_1(s)}{1+K(s)G_2(s)} = \frac{(K_d(s)G_{d,1}(s)+K_n(s)G_{n,1}(s))}{(K_d(s)G_{d,2}(s)+K_n(s)G_{n,2}(s))}\frac{G_{d,2}(s)}{G_{d,1}(s)}. \quad (12)$$

It should be noted that $K_d(s)G_{d,1}(s)+K_n(s)G_{n,1}(s)$ is the characteristic polynomial of one of the stable closed-loop system thus, it can be replaced by $c_1(s)$ and $K_d(s)G_{d,2}(s)+K_n(s)G_{n,2}(s)$ is the characteristic polynomial of the other stable closed-loop system and can be replaced by $c_2(s)$. Thus, (12) can be written as:

$$\frac{1+K(s)G_1(s)}{1+K(s)G_2(s)} = \frac{c_1(s)}{c_2(s)}\frac{G_{d,2}(s)}{G_{d,1}(s)}. \quad (13)$$

By simple manipulations we get:

$$\frac{c_1(s)}{c_2(s)} = \frac{1+K(s)G_1(s)}{1+K(s)G_2(s)}\frac{G_{d,1}(s)}{G_{d,2}(s)}. \quad (14)$$

Using the second property of Theorem 1, the closed-loop system is quadratically stable iff:

$$|\arg(1+K(j\omega)G_1(j\omega)) - \arg(1+K(j\omega)G_2(j\omega)) + \arg(G_{d,1}(j\omega)) - \arg(G_{d,2}(j\omega))| < \frac{\pi}{2} \quad \forall\omega. \quad (15)$$

Replacing $\arg(G_{d,1}(j\omega)) - \arg(G_{d,2}(j\omega))$ by $\Delta(\omega)$, we have:

$$-\frac{\pi}{2} - \Delta(\omega) < \arg(1+K(j\omega)G_1(j\omega)) - \arg(1+K(j\omega)G_2(j\omega)) < \frac{\pi}{2} - \Delta(\omega) \quad \forall\omega. \quad (16)$$

It should be noted that $\Delta(\omega)$ is known since the parametric models of the systems are known. These inequalities can be simply transformed into linear constraints in the Nyquist diagram.

Let us consider the simple case when $\Delta(\omega)$ is zero for all frequencies (the denominators of the two models are equal), then (16) becomes:

$$|\arg(1+K(j\omega)G_1(j\omega)) - \arg(1+K(j\omega)G_2(j\omega))| < \frac{\pi}{2} \, \forall\omega. \quad (17)$$

In this case, the idea is to add to the line d_1 assuring the robust stability, two perpendicular lines d_u and d_l passing through the -1 point to assure the quadratic stability (see Fig. 2). If $K(j\omega)G_1(j\omega)$ and $K(j\omega)G_2(j\omega)$ are between these two lines, γ, the difference between the arguments of $1+K(j\omega)G_1(j\omega)$ and $1+K(j\omega)G_2(j\omega)$, is always less than $\pi/2$, thus respecting Inequality (17). By the way, it should be noted that Fig. 2 is a good illustration of the restrictions added by the quadratic stability. Indeed, the region where $K(j\omega)G_1(j\omega)$ and $K(j\omega)G_2(j\omega)$ are allowed

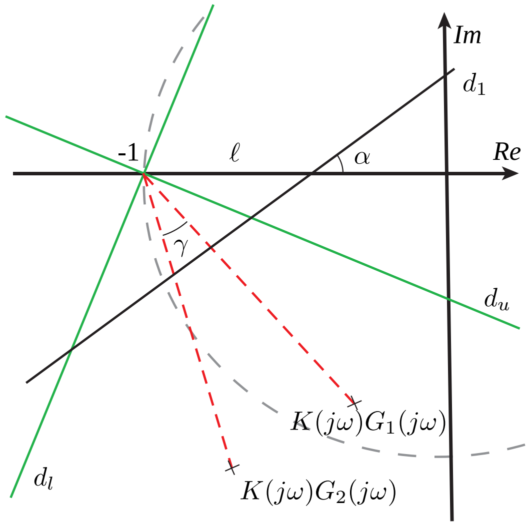

Fig. 2. Linear constraints for quadratic stability when $\Delta(\omega) = 0$ and for robust stability (d_1).

to be to assure robust and quadratic stability is smaller than the region assuring only robust stability.

Since the open-loop transfer functions are strictly proper, it is impossible to respect the constraints defined by d_u and d_l at high frequency as depicted in Fig. 2. At high frequency, the region defined by d_u and d_l should include the origin of the Nyquist diagram. This problem is solved by defining the lines d_u and d_l as a function of the frequency. To do it, a desired open-loop transfer function $L_d(s)$ is introduced. For each frequency, d_u and d_l are defined such that the vector $1+L_d(j\omega)$ is the angle bisector of the two lines. In other words, d_l and d_u have a phase difference of, respectively, $-\pi/4$ and $\pi/4$ with respect to $1+L_d(j\omega)$.

This idea works for the simple case when $\Delta(\omega)$ is zero but, for the more general case when $\Delta(\omega)$ is different from zero, $K(j\omega)G_1(j\omega)$ and $K(j\omega)G_2(j\omega)$ have to be located in different regions. Examining Inequality (16), these regions should have an offset of $\Delta(\omega)$ between them. This is why $K(j\omega)G_1(j\omega)$ should be between the lines $d_{l,1}$ and $d_{u,1}$ ($d_{l,1}$ and $d_{u,1}$ have a phase difference of, respectively, $-\pi/4 - \Delta(\omega)/2$ and $\pi/4 - \Delta(\omega)/2$ with respect to $1+L_d(j\omega)$) and $K(j\omega)G_2(j\omega)$ between the lines $d_{l,2}$ and $d_{u,2}$ ($d_{l,2}$ and $d_{u,2}$ have a phase difference of, respectively, $-\pi/4+\Delta(\omega)/2$ and $\pi/4+\Delta(\omega)/2$ with respect to $1+L_d(j\omega)$) (see Fig. 3).

The line $d_{u,1}$ can be described by:

$$y - m_{u,1}(\omega)x - m_{u,1}(\omega) = 0 \quad (18)$$

where x and y are, respectively, the coordinates of the real and imaginary axes and $m_{u,1}(\omega)$ the slope of $d_{u,1}$ at the frequency ω. When $-3\pi/4 < \Delta(\omega)/2 < \pi/4$, $K(j\omega)G_1(j\omega)$ and $L_d(j\omega)$ must be on the same side of $d_{u,1}$. Thus, this can be expressed by the following linear constraint:

$$\begin{aligned}&(\mathcal{I}_{L_d}(\omega) - m_{u,1}(\omega)\mathcal{R}_{L_d}(\omega) - m_{u,1}(\omega))\\ &(\rho^T(\mathcal{I}_1(\omega) - m_{u,1}(\omega)\mathcal{R}_1(\omega)) - m_{u,1}(\omega)) \geq 0\end{aligned} \quad (19)$$

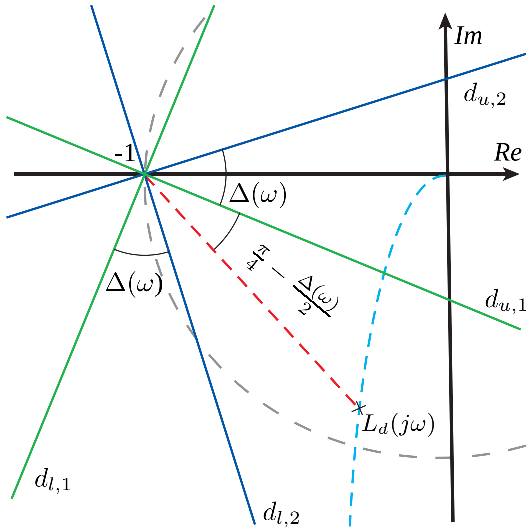

Fig. 3. Linear constraints to ensure quadratic stability in the Nyquist diagram.

where $\mathcal{I}_{L_d}(\omega)$ and $\mathcal{R}_{L_d}(\omega)$ are, respectively, the imaginary and real parts of $L_d(j\omega)$. By examining Fig. 3, we can notice that, if $\pi/4 < \Delta(\omega)/2 < 5\pi/4$, $K(j\omega)G_1(j\omega)$ and $L_d(j\omega)$ must not be on the same side of $d_{u,1}$, so $\geq$ in (19) should be replaced by $\leq$. For the sake of simplicity, we define the following function:

$$f_1(\Delta(\omega)) = \text{sgn}\left(\cos\left(\frac{\Delta(\omega)}{2}+\frac{\pi}{4}\right)\right) \tag{20}$$

which is negative when $\pi/4 < \Delta(\omega)/2 < 5\pi/4$. Thus, the following linear constraint is defined whatever the value of $\Delta(\omega)/2$ is:

$$\begin{aligned} f_1(\Delta(\omega))[\mathcal{I}_{L_d}(\omega) - m_{u,1}(\omega)\mathcal{R}_{L_d}(\omega) - m_{u,1}(\omega)] \\ [\rho^T(\mathcal{I}_1(\omega) - m_{u,1}(\omega)\mathcal{R}_1(\omega)) - m_{u,1}(\omega)] \geq 0 \end{aligned} \tag{21}$$

For similar reason we define:

$$f_2(\Delta(\omega)) = \text{sgn}\left(\cos\left(\frac{\Delta(\omega)}{2}-\frac{\pi}{4}\right)\right) \tag{22}$$

Thus, linear constraints can be written for lines $d_{l,1}$, $d_{u,2}$ and $d_{l,2}$:

$$\begin{aligned} f_2(\Delta(\omega))[\mathcal{I}_{L_d}(\omega) - m_{l,1}(\omega)\mathcal{R}_{L_d}(\omega) - m_{l,1}(\omega)] \\ [\rho^T(\mathcal{I}_1(\omega) - m_{l,1}(\omega)\mathcal{R}_1(\omega)) - m_{l,1}(\omega)] \geq 0 \end{aligned} \tag{23}$$

$$\begin{aligned} f_2(\Delta(\omega))[\mathcal{I}_{L_d}(\omega) - m_{u,2}(\omega)\mathcal{R}_{L_d}(\omega) - m_{u,2}(\omega)] \\ [\rho^T(\mathcal{I}_2(\omega) - m_{u,2}(\omega)\mathcal{R}_2(\omega)) - m_{u,2}(\omega)] \geq 0 \end{aligned} \tag{24}$$

$$\begin{aligned} f_1(\Delta(\omega))[\mathcal{I}_{L_d}(\omega) - m_{l,2}(\omega)\mathcal{R}_{L_d}(\omega) - m_{l,2}(\omega)] \\ [\rho^T(\mathcal{I}_2(\omega) - m_{l,2}(\omega)\mathcal{R}_2(\omega)) - m_{l,2}(\omega)] \geq 0 \end{aligned} \tag{25}$$

where $m_{l,1}(\omega)$, $m_{u,2}(\omega)$ and $m_{l,2}(\omega)$ are, respectively, the slopes of $d_{l,1}$, $d_{u,2}$ and $d_{l,2}$ at the frequency ω. It should be noted that when one of these lines is vertical at a particular frequency, the slope becomes infinity. Thus, the inequalities given above should not be used. For example, (21) should be replaced by:

$$f_1(\Delta(\omega))[\mathcal{R}_{L_d}(\omega)+1][\rho^T\mathcal{R}_1(\omega)+1] \geq 0. \tag{26}$$

The choice of $L_d(s)$ plays a crucial role since the constraints depend on it. There are some simple choices that usually leads to good results for simple models. For example $L_d = w_c/s$ is an appropriate choice for low-order stable systems.

Finally, these new constraints should simply be added to the optimization problems defined in [1]. For this purpose, a sufficient number of frequencies should be chosen between 0 and infinity. For example, when the quadratic stability is needed, the optimization problem (8) becomes:

$$\text{maximize } k_0$$

$$\begin{aligned} &\text{subject to:} \\ &\rho^T(\cot\alpha\,\mathcal{I}_i(\omega_k) - \mathcal{R}_i(\omega_k)) + \ell \leq 1 \quad \forall\omega_k,\ i=1,2. \\ &f_1(\Delta(\omega_k))[\mathcal{I}_{L_d}(\omega_k) - m_{u,1}(\omega_k)\mathcal{R}_{L_d}(\omega_k) - m_{u,1}(\omega_k)] \\ &[\rho^T(\mathcal{I}_1(\omega_k) - m_{u,1}(\omega_k)\mathcal{R}_1(\omega_k)) - m_{u,1}(\omega_k)] \geq 0 \quad \forall\omega_k \\ &f_2(\Delta(\omega_k))[\mathcal{I}_{L_d}(\omega_k) - m_{l,1}(\omega_k)\mathcal{R}_{L_d}(\omega_k) - m_{l,1}(\omega_k)] \\ &[\rho^T(\mathcal{I}_1(\omega_k) - m_{l,1}(\omega_k)\mathcal{R}_1(\omega_k)) - m_{l,1}(\omega_k)] \geq 0 \quad \forall\omega_k \\ &f_2(\Delta(\omega_k))[\mathcal{I}_{L_d}(\omega_k) - m_{u,2}(\omega_k)\mathcal{R}_{L_d}(\omega_k) - m_{u,2}(\omega_k)] \\ &[\rho^T(\mathcal{I}_2(\omega_k) - m_{u,2}(\omega_k)\mathcal{R}_2(\omega_k)) - m_{u,2}(\omega_k)] \geq 0 \quad \forall\omega_k \\ &f_1(\Delta(\omega_k))[\mathcal{I}_{L_d}(\omega_k) - m_{l,2}(\omega_k)\mathcal{R}_{L_d}(\omega_k) - m_{l,2}(\omega_k)] \\ &[\rho^T(\mathcal{I}_2(\omega_k) - m_{l,2}(\omega_k)\mathcal{R}_2(\omega_k)) - m_{l,2}(\omega_k)] \geq 0 \quad \forall\omega_k \end{aligned} \tag{27}$$

In this section, the theory is developed to guarantee the quadratic stability of multi-model systems composed of two models. It should be noted that this method can also be applied to the following class of systems:

- LPV systems with affine dependency on one scheduling parameter can be quadratically stabilized with a fixed controller by the proposed approach. It is sufficient to impose the quadratic stability constraints on the two open-loop systems corresponding to the extremities of the range of the scheduling parameter to ensure global stability.
- LPV systems with affine dependency on one scheduling parameter only in the denominator can be quadratically stabilized with an LPV controller with affine dependency on one scheduling parameter only in the numerator. Once again, it is sufficient to impose the quadratic stability constraints on the two open-loop systems corresponding to the extremities of the range of the scheduling parameter to ensure global stability.
- Switched systems composed of two subsystems. It is possible to design a specific controller for each subsystem. The only constraint is that the two controllers should have the same basis functions. This is illustrated in the following section.

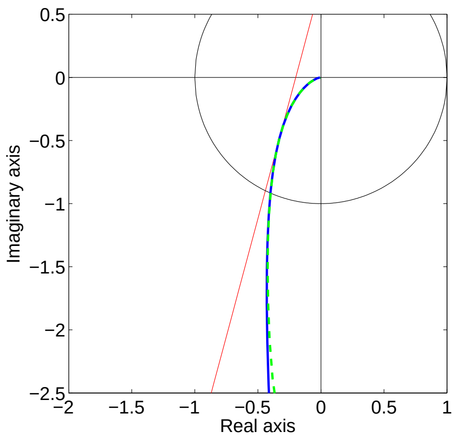

Fig. 4. Nyquist plots of the open-loop transfer functions $K_1(j\omega)G_1(j\omega)$ (solid) and $K_2(j\omega)G_2(j\omega)$ (dashed).

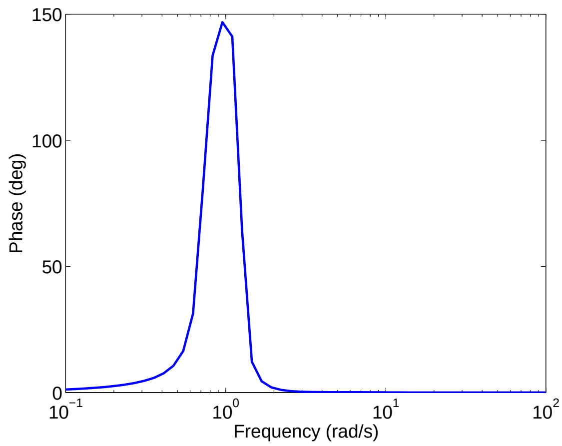

Fig. 5. Phase difference between the two characteristic polynomials of the closed-loop subsystems.

IV. Simulation Results

To show the effectiveness of the method, it is applied to the following switched system composed of two subsystems:

$$G_1(s) = \frac{0.5}{s^2 + 0.2s + 0.5} \quad , \quad G_2(s) = \frac{1.5}{s^2 + 0.2s + 1.5}.$$

The objective is to design a PID controller with the following structure:

$$K(s) = \frac{K_d s^2 + K_p s + K_i}{s(1 + Ts)} \tag{28}$$

for each subsystem that maximizes the load disturbance rejection. Two cases are treated: first, two controllers will be designed without taking care of quadratic stability, then two controllers will be designed with the additional linear constraints, thus assuring quadratic stability.

For the first case, the linear programming approach, that maximizes the load disturbance rejection is used to design the two controllers. The design variables ℓ is set to 0.8 and α to 75° to get a good modulus margin and a well-damped closed loop-system. The time constant T of the filter of the PID controller is set to 0.1. To be able to use the proposed method, the subsystems $G_1(s)$ and $G_2(s)$ are evaluated at 50 logarithmically spaced points between 0.1 and 100 rad/s. The Nyquist plots of the open-loop transfer functions of the two subsystems with the two controllers are shown in Fig. 4. It can be observed that the Nyquist plots respect the linear robustness constraint (red line). Finally, the phase difference between the two characteristic polynomials of the two closed-loop subsystems is shown in Fig. 5. It can be observed that for certain frequencies, the phase difference is greater than $\pi/2$. This means that the switched system composed of these two subsystems is not quadratically stable. This is confirmed by the fact that no common Lyapunov function could be found for the two subsystems.

For the second case, the same optimization problem is used with the difference that the constraints to ensure the quadratic stability are added. To add these constraints, a desired open-loop transfer function $L_d(s)$ has to be chosen. $L_d(s)$ is chosen equal to $\omega_c/(s(1+Ts))$ where ω_c is equal to 2.5 rad/s. This value is the largest for which the optimization problem is feasible.

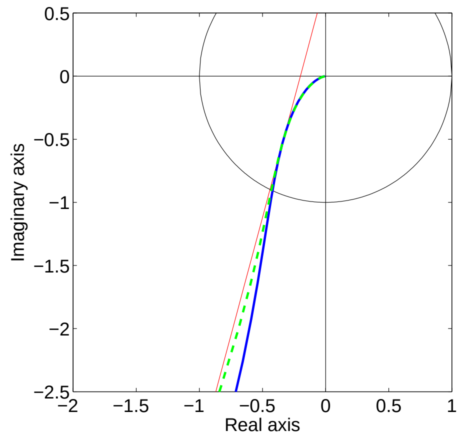

Fig. 6. Nyquist plots of the open-loop transfer functions $K_1(j\omega)G_1(j\omega)$ (solid) and $K_2(j\omega)G_2(j\omega)$ (dashed) guaranteeing quadratic stability.

The Nyquist plots of the open-loop transfer functions of the two subsystems with the two controllers are shown in Fig. 6. It can be observed that the Nyquist plots respect the linear robustness constraint (red line). Finally, the phase difference between the two characteristic polynomials of the two closed-loop subsystems is shown in Fig. 7. It can be observed that for all the frequencies, the phase is smaller than $\pi/2$ in absolute value. It means that the switched system is globally stable. This is confirmed by the fact that a common Lyapunov function could be found for the two subsystems.

V. Conclusions

In this paper, an extension of the frequency method based on linear programming is proposed to design fixed-order linearly parameterized controllers for multi-model systems composed of two models guaranteeing quadratic stability.

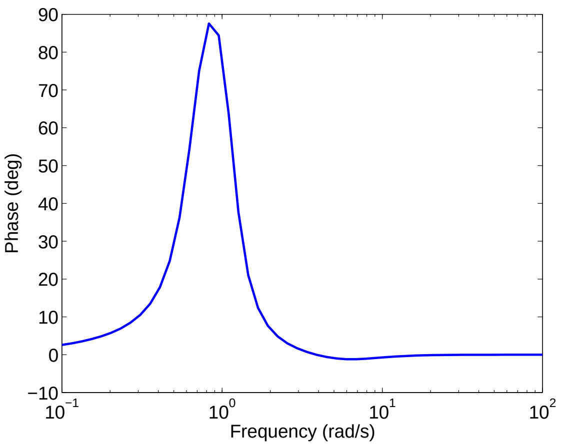

Fig. 7. Phase difference between the two characteristic polynomials of the closed-loop subsystems guaranteeing quadratic stability.

The method is based on frequency loop shaping in the Nyquist diagram. Classical robustness and performance specifications are represented by linear constraints. The quadratic stability is ensured by adding linear constraints based on the phase difference of the two characteristic polynomials of the closed-loop systems. This method can also be applied to switched systems composed of two subsystems and to LPV systems whose scheduling parameter has an affine dependency in the closed-loop expression. Future work consists of extending this method to multi-model systems composed of more than two models.

References

[1] A. Karimi, M. Kunze, and R. Longchamp, "Robust controller design by linear programming with application to a double-axis positioning system," *Control Engineering Practice*, vol. 15, no. 2, pp. 197–208, February 2007.

[2] Y. Chait, Q. Chen, and C. V. Hollot, "Automatic loop-shaping of QFT controllers via linear programming," *Journal of Dynamic Systems, Measurement and Control*, vol. 121, no. 3, pp. 351–357, September 1999.

[3] M. Kunze, A. Karimi, and R. Longchamp, "Gain-scheduled controller design by linear programming," in *European Control Conference*, July 2007, pp. 5432–5438.

[4] W. J. Rugh and J. S. Shamma, "Research on gain scheduling," *Automatica*, vol. 36, no. 10, pp. 1401–1425, October 2000.

[5] D. J. Leith and W. E. Leithead, "Survey of gain-scheduling analysis and design," *International Journal of Control*, vol. 73, no. 11, pp. 1001–1025, July 2000.

[6] P. Ioannou and G. Tao, "Frequency domain conditions for strictly positive real functions," *IEEE Transactions on Automatic Control*, vol. 32, no. 1, pp. 53–54, January 1987.

[7] K. Zhou, *Essentials of Robust Control*. New Jersey: Prentice Hall, 1998.

Appendix

A. Proof of Theorem 1

Proof: (1) ⇔ (2): This equivalence is immediate. For more details see [6].

(1) ⇒ (3): Consider the transfer function $\frac{c_1(s)}{c_2(s)}$:

$$\frac{c_1(s)}{c_2(s)} = 1 + \frac{c_1(s) - c_2(s)}{c_2(s)}. \tag{29}$$

Using a controllable canonical form, (29) leads to the following state space realization:

$$\left[\begin{array}{c|c} A_2 & B \\ \hline C_1 - C_2 & 1 \end{array}\right] \tag{30}$$

with $B = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}^T$.

Using the KYP lemma, (30) is SPR iff:

$$\exists P = P^T > 0 \; s.t. \\ \begin{bmatrix} A_2^T P + P A_2 & PB - (C_1 - C_2)^T \\ B^T P - (C_1 - C_2) & -2 \end{bmatrix} < 0. \tag{31}$$

Using the Schur lemma, we have:

$$A_2^T P + P A_2 + \frac{1}{2}[PB - (C_1 - C_2)^T][B^T P - (C_1 - C_2)] < 0. \tag{32}$$

Adding and substracting $\frac{1}{2}PB(C_1 - C_2) + \frac{1}{2}(C_1 - C_2)^T B^T P$ to (32), we get:

$$(A_2 - B(C_1 - C_2))^T P + P(A_2 - B(C_1 - C_2)) + \\ \frac{1}{2}(PB + (C_1 - C_2)^T)(B^T P + (C_1 - C_2)) < 0. \tag{33}$$

Knowing that $A_1 = A_2 - B(C_1 - C_2)$ gives:

$$A_1^T P + P A_1 + \\ \frac{1}{2}(PB - (C_2 - C_1)^T)(B^T P - (C_2 - C_1)) < 0. \tag{34}$$

Since the third terms in (32) and (34) are positive semi-definite, we obtain $A_1^T P + P A_1 < 0$ and $A_2^T P + P A_2 < 0$. This proves that A_1 and A_2 are quadratically stable.

(3) ⇒ (1): This equivalence is proved using the bounded real lemma. If A_1 and A_2 are quadratically stable, it means that the following matrix is stable:

$$\frac{A_1 + A_2}{2} + \gamma B \frac{C_2 - C_1}{2} \tag{35}$$

for all values of γ between -1 and 1. According to the bounded real lemma we have [7]:

$$\left(\frac{A_1 + A_2}{2}\right)^T P + P\left(\frac{A_1 + A_2}{2}\right) + PBB^T P + \\ \left(\frac{C_2 - C_1}{2}\right)^T \left(\frac{C_2 - C_1}{2}\right) < 0. \tag{36}$$

It is well known that the bounded real lemma is equivalent to the positive real lemma with:

$$\bar{A} = A - BC = \frac{A_1 + A_2}{2} - B\frac{C_2 - C_1}{2} = A_2,$$

$\bar{B} = B$, $\bar{C} = -2C = C_1 - C_2$, $\bar{D} = 1$ and $\bar{P} = 2P$. Thus, we get:

$$\exists \bar{P} = \bar{P}^T > 0 \; s.t. \begin{bmatrix} \bar{A}^T \bar{P} + \bar{P}\bar{A} & \bar{P}\bar{B} - \bar{C}^T \\ \bar{B}^T \bar{P} - \bar{C} & -\bar{D} - \bar{D}^T \end{bmatrix} = \\ \begin{bmatrix} A_2^T \bar{P} + \bar{P} A_2 & \bar{P}B - (C_1 - C_2)^T \\ B^T \bar{P} - (C_1 - C_2) & -2 \end{bmatrix} < 0. \tag{37}$$

This inequality is the same as (31) thus, we can conclude that $\frac{c_1(s)}{c_2(s)}$ is SPR. Using the properties of SPR systems, the inverse, $\frac{c_2(s)}{c_1(s)}$, is also SPR. ■