

AN UNEXPECTED PROPERTY OF ODD ORDER DERIVATIVES OF HARDY'S FUNCTION

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ABSTRACT. Assuming the Riemann hypothesis, we show that the odd order derivatives of Hardy's function have, under some condition, an unexpected behavior for large values of t .

1. Introduction and main result

Let ζ be the Riemann zeta function, and Z the Hardy function defined by

$$Z(t) = e^{i\theta(t)} \zeta\left(\frac{1}{2} + it\right)$$

where

$$\theta(t) = \arg\left(\pi^{-i\frac{t}{2}} \Gamma\left(\frac{1}{4} + i\frac{t}{2}\right)\right)$$

and the argument is defined by continuous variation of t starting with the value 0 at $t = 0$. It can be shown [6] that

$$\theta(t) = \frac{t}{2} \log \frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} + O\left(\frac{1}{t}\right).$$

The real zeros of Z coincide with the zeros of ζ located on the line of real part $\frac{1}{2}$. If the Riemann hypothesis is true, then the number of zeros of Z in the interval $]0, t]$ is given by [6]

$$(1.1) \quad N(t) = \frac{1}{\pi} \theta(t) + 1 + S(t)$$

where $S(t) = \frac{1}{\pi} \arg \zeta\left(\frac{1}{2} + it\right)$ if t is not a zero of Z and $\arg \zeta\left(\frac{1}{2} + it\right)$ is defined by continuous variation along the straight lines joining 2 , $2 + it$ and $\frac{1}{2} + it$ starting

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Dedicated to Professor Jean Descloux on the occasion of his 80th birthday.

with the initial value $\arg \zeta(2) = 0$. If t is a zero of Z we set $S(t) = \lim_{\epsilon \rightarrow 0+} S(t + \epsilon)$. Let us choose T , say $T = 500$, and let us plot the graphs of functions

$$f_{2k-1}(t) := (-1)^{k+1} \frac{Z^{(2k-1)}(t)}{\theta'(T)^{2k-1}}$$

for $k = 1, \dots, 5$ on the interval $[T - 10, T + 10]$. Observe that the term $1/\theta'(T)^{2k-1}$ is just a scaling factor. These graphs show that the functions $(-1)^{k+1} Z^{(2k-1)}$ have

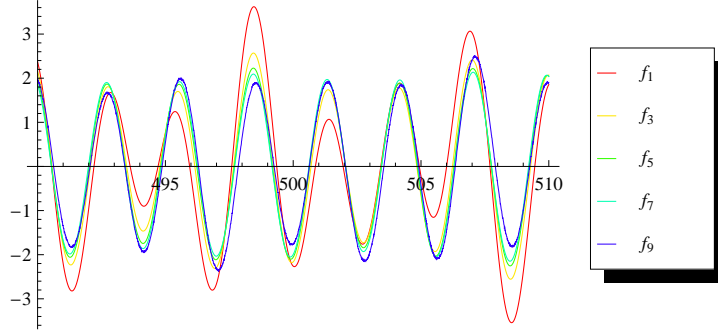


FIGURE 1. Graphs of f_{2k-1} for $k = 1, \dots, 5$

generally the same signs, at least for small values of k . This can be explained heuristically by a formula due to Lavrik [8], which asserts that for t sufficiently large and $1 \leq k \leq \frac{1}{4} \log t$, we have, uniformly in k ,

$$Z^{(2k-1)}(t) = 2(-1)^k \sum_{1 \leq n \leq \sqrt{t/2\pi}} \frac{1}{\sqrt{n}} (\theta'(t) - \log n)^{2k-1} \sin(\theta(t) - t \log n) + O(t^{-\frac{1}{4}} (\frac{3}{2} \log t)^{2k}).$$

Let us denote by $M = M(T)$ the largest integer r , possibly infinite, such that

$$\text{sign}((-1)^{k+1} Z^{(2k-1)}(T)) = \text{sign}(Z'(T)) \quad \text{for } k = 1, 2, \dots, r.$$

For some T the values of M are surprisingly large. Using Mathematica we compute Z with high precision and get for example $M(100) = 26$, $M(1000.4) = 138$ and $M(9999.5) = 402$.

Now let T be large enough such that $Z(T) > 0$ and let γ_k , where $k \neq 0$, be the zeros of Z ordered in increasing order, taking their multiplicities into account, and numbered so that $\dots \leq \gamma_{-2} \leq \gamma_{-1} < T < \gamma_1 \leq \gamma_2 \leq \dots$. Further, let $4(\log \log T)^{-1} \leq a \leq \sqrt{T}$ such that $T + a$ and $T - a$ are not zeros of Z and, finally, let $m, n \geq 1$ such that $\gamma_{-m-1} < T - a < \gamma_{-m}$ and $\gamma_n < T + a < \gamma_{n+1}$. Note that the existence of $m, n \geq 1$ is an immediate consequence of a result of Goldston and Gonek [4]. We assume that $(-1)^m Z'(T - a) \geq 0$ and $(-1)^n Z'(T + a) \leq 0$ and we

denote by $K = K(T, a)$ the largest integer r , possibly infinite, such that

$$(1.2) \quad (-1)^{m+k+1} Z^{(2k-1)}(T-a) \geq 0 \quad \text{and} \quad (-1)^{n+k+1} Z^{(2k-1)}(T+a) \leq 0$$

for $k = 1, 2, \dots, r$. For some T and a , the values of K are also surprisingly large: $K(109.3, 9.4) = 21$, $K(1070.1, 8.5) = 108$ and $K(10025.5, 9.8) = 408$. The goal of this paper is to give a conditional upper bound for K .

It should be observed that if we replace $Z(t)$ by $\cos t$ and choose $T = 0$ and $a = l\pi + \frac{\pi}{4}$ where $l \in \mathbb{N}^*$, then $m = n = l$ and conditions (1.2) hold for every k and hence $K = \infty$.

We now define a quantity which appears in our main result. By Lavrik's formula [8], for t sufficiently large and $0 \leq k \leq \frac{1}{4} \log t$, we have, uniformly in k ,

$$Z^{(2k)}(t) = 2(-1)^k \sum_{1 \leq n \leq \sqrt{t/2\pi}} \frac{1}{\sqrt{n}} (\theta'(t) - \log n)^{2k} \cos(\theta(t) - t \log n) + O(t^{-\frac{1}{4}} (\frac{3}{2} \log t)^{2k+1})$$

and using $\theta'(t) = \frac{1}{2} \log \frac{t}{2\pi} + O(1/t^2)$ we get

$$(1.3) \quad |Z^{(2k)}(t)| = \theta'(t)^{2k} \left(2 \sum_{1 \leq n \leq \sqrt{t/2\pi}} \frac{1}{\sqrt{n}} \left(1 - \frac{\log n}{\theta'(t)} \right)^{2k} \cos(\theta(t) - t \log n) \right) + O(3^{2k} \theta'(t)^{2k} t^{-\frac{1}{4}} \log t).$$

As Ivić says in [5], it is difficult to get good uniform bounds for $Z^{(2k)}(t)$ from (1.3). Nevertheless, when $Z(T)$ is large, relation (1.3) suggests that

$$(1.4) \quad \int_{T-a}^{T+a} (Z^{(2k)}(t))^2 dt = 2a (c_{2k} \theta'(T)^{2k} Z(T))^2$$

where c_{2k} is small. For the aforementioned computations, we used the results of Kotnik [7] and chose T in a neighborhood of 10^2 , 10^3 and 10^4 and a approximately equal to 10 such that $Z(T)$ and $K(T, a)$ are large. We obtain $c_{42} = 0.56 \dots$ and $c_{216} = 0.34 \dots$ which show that c_{2k} can be small even for some $k > \frac{1}{4} \log T$ despite the fact that, for fixed T , the sequence c_{2k} is unbounded. Numerical experiments indicate that Lavrik's formula is probably true for larger values of k with a better error term.

THEOREM 1.1. *For T large enough, let $4(\log \log T)^{-1} \leq a \leq \sqrt{T}$, K be the number defined in the introduction, $\Delta S = S(T+a) - S(T-a)$ and further let $K^* = \frac{a}{\pi} \log T + \Delta S$. If the Riemann hypothesis is true and if $\Delta S \geq 1$, then*

$$(1.5) \quad K \leq \max \left(\frac{a}{2\pi} A_{K,T} \frac{\log T}{\Delta S} \left(1 + O \left(\frac{\log \log \log T}{\log \log T} \right) \right), K^* \log K^* \right)$$

where

$$A_{K,T} = \log c_{2K} + \Delta S \log 2 + \frac{1}{\sqrt{2}} \frac{\log T}{\log \log T}.$$

For T large enough such that $Z(T)$ is large in the sense of [7], numerical experiments show that the bound (1.5) is probably true without the term $\log c_{2K}$. If this is the case and if we neglect the big O in (1.5) and choose $T \leq 10^{50}$ and $a \leq 1$ such that $\Delta S = 1$, we get $K \leq 327$. Note that for the three values of $K(T, a)$ already given, we have $\Delta S < 0$. This suggests that the behaviour of K is different according to $\Delta S \leq 0$ or $\Delta S \geq 1$. This is unexpected.

This work stems from an observation of Ivić [5] about the values of the derivatives of Z in a neighborhood of points where $|Z|$ attains a large value. Some of the material used in our proof has been published by the author in [1].

The notations used in this paper are standard: $\lfloor x \rfloor$ stand for the usual floor function and $\{x\} := x - \lfloor x \rfloor$. Bernoulli and Chebyshev polynomials of degree n are denoted by $B_n(x)$ and $T_n(x)$; they are defined by

$$\int_x^{x+1} B_n(t) dt = x^n \quad \text{and} \quad T_n(\cos \theta) = \cos n\theta.$$

The organization of this paper is as follows: In Section 2 we prove the key identity, a property of the derivatives of Bernoulli polynomials and preparatory lemmas. Section 3 is devoted to the proof of our main result.

2. Preliminary results

We first prove an identity which will be used later to establish a relation between the value of a function $f \in C^{2r}[-a, a]$ at 0, the zeros of f and the values of its derivatives of odd order on the boundaries of the interval.

LEMMA 2.1. *Let $-a < x_{-m} < \dots < x_{-1} < x_0 < x_1 < \dots < x_n < a$ and for $l = 1, 2, \dots$ let Ψ_{2l-1} be the function defined on $[-a, a]$ by*

$$\Psi_{2l-1}(x) = \frac{(4a)^{2l-1}}{(2l)!} \sum_{k=-m}^n \mu_k \left(B_{2l} \left(\frac{1}{2} + \frac{x+x_k}{4a} \right) + B_{2l} \left(\left\{ \frac{x-x_k}{4a} \right\} \right) \right)$$

where $\sum_{k=-m}^n \mu_k = 0$. Then for $f \in C^{2r}[-a, a]$ where $r \geq 1$, we have the identity

$$(2.1) \quad \sum_{k=-m}^n \mu_k f(x_k) = \sum_{k=1}^r f^{(2k-1)}(a) \Psi_{2k-1}(a) - \sum_{k=1}^r f^{(2k-1)}(-a) \Psi_{2k-1}(-a) - \int_{-a}^a f^{(2r)}(x) \Psi_{2r-1}(x) dx.$$

PROOF. By definition the function Ψ_{2r-1} is C^{2r-2} , piecewise polynomial and the relation $B'_l(x) = lB_{l-1}(x)$ for $l = 1, 2, \dots$ leads to

$$\Psi_{2r-1}^{(j)}(x) = \frac{(4a)^{2r-j-1}}{(2r-j)!} \sum_{k=-m}^n \mu_k \left(B_{2r-j} \left(\frac{1}{2} + \frac{x+x_k}{4a} \right) + B_{2r-j} \left(\left\{ \frac{x-x_k}{4a} \right\} \right) \right)$$

for $j = 1, \dots, 2r-1$ and $x \neq x_{-m}, \dots, x_n$ if $j = 2r-1$. This implies that

$$(2.2) \quad \Psi_{2r-1}^{(2r-2j)} = \Psi_{2j-1} \quad \text{for } j = 1, 2, \dots, r-1$$

and that

$$\begin{aligned}
(2.3) \quad \Psi_{2r-1}^{(2r-2j+1)}(\pm a) &= \frac{(4a)^{2j-2}}{(2j-1)!} \sum_{k=-m}^n \mu_k \left(B_{2j-1} \left(\frac{1}{2} + \frac{\pm a + x_k}{4a} \right) + B_{2j-1} \left(\left\{ \frac{\pm a - x_k}{4a} \right\} \right) \right) \\
&= \frac{(4a)^{2j-2}}{(2j-1)!} \sum_{k=-m}^n \mu_k \left(B_{2j-1} \left(\frac{1}{2} + \frac{\pm a + x_k}{4a} \right) + B_{2j-1} \left(\frac{1}{2} - \frac{\pm a + x_k}{4a} \right) \right) \\
&= 0 \quad \text{for } j = 1, 2, \dots, r.
\end{aligned}$$

Further for $x \neq x_{-m}, \dots, x_n$ we have

$$\begin{aligned}
(2.4) \quad \Psi_{2r-1}^{(2r-1)}(x) &= \sum_{k=-m}^n \mu_k \left(B_1 \left(\frac{1}{2} + \frac{x + x_k}{4a} \right) + B_1 \left(\left\{ \frac{x - x_k}{4a} \right\} \right) \right) \\
&= \sum_{k=-m}^n \mu_k \left(\frac{x + x_k}{4a} + \left\{ \frac{x - x_k}{4a} \right\} - \frac{1}{2} \right)
\end{aligned}$$

and as $\sum_{k=-m}^n \mu_k = 0$ the function $\Psi_{2r-1}^{(2r-1)}$ is piecewise constant. Explicitly, for $x \in]x_j, x_{j+1}[$, we get

$$\Psi_{2r-1}^{(2r-1)}(x) = \sum_{k=-m}^j \mu_k \left(\frac{x}{2a} - \frac{1}{2} \right) + \sum_{k=j+1}^n \mu_k \left(\frac{x}{2a} + 1 - \frac{1}{2} \right) = \sum_{k=j+1}^n \mu_k = - \sum_{k=-m}^j \mu_k$$

which leads to

$$\int_{x_j}^{x_{j+1}} f'(x) \Psi_{2r-1}^{(2r-1)}(x) dx = - \left(\sum_{k=-m}^j \mu_k \right) (f(x_{j+1}) - f(x_j)) \quad \text{for } j = -m, \dots, n-1.$$

Summing these equalities and using that $\Psi_{2r-1}^{(2r-1)} = 0$ on the intervals $[-a, x_{-m}[$ and $]x_n, a]$, which follows from (2.4), we have

$$\sum_{k=-m}^n \mu_k f(x_k) = \int_{-a}^a f'(x) \Psi_{2r-1}^{(2r-1)}(x) dx$$

and we complete the proof by integrating $2r-1$ times the right-hand side by parts taking into account relations (2.2) and (2.3). \square

For further use we recall some elementary facts concerning the divided differences.

LEMMA 2.2. *Let $I =]-T, T[$, $f \in C^{m+n}(I)$ and let g be the function defined for pairwise distinct numbers $t_{-m}, \dots, t_n \in I$ by*

$$g(t_{-m}, \dots, t_n) = \sum_{k=-m}^n \frac{f(t_k)}{\prod_{\substack{-m \leq j \leq n \\ j \neq k}} (t_k - t_j)}.$$

Then

- a) The function g has a continuous extension g^* defined for $t_{-m}, \dots, t_n \in I$ and there exists $\eta = \eta(t_{-m}, \dots, t_n) \in I$ such that

$$g^*(t_{-m}, \dots, t_n) = \frac{f^{(m+n)}(\eta)}{(m+n)!}.$$

- b) Let y_0, y_1, \dots, y_l be the distinct values of t_{-m}, \dots, t_n considered as fixed and let r_k be the number of index j such that $t_j = y_k$. Then there exist $\alpha_{k,i}$ depending on y_0, y_1, \dots, y_l such that

$$g^*(t_{-m}, \dots, t_n) = \sum_{k=0}^l \sum_{i=0}^{r_k-1} \alpha_{k,i} f^{(i)}(y_k).$$

PROOF. Assertion a) is a consequence of the representation formula

$$(2.5) \quad g(t_{-m}, \dots, t_n) = \int_0^1 d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{m+n-1}} f^{(m+n)} \left(t_{-m} + \sum_{k=1}^{m+n} \tau_k (t_{-m+k} - t_{-m+k-1}) \right) d\tau_{m+n}.$$

A proof of b) is given in [9]. □

In the next lemma we indicate the choice of coefficients μ_k for which the identity of Lemma 2.1 is of practical use for large values of a . The main reason of this choice will appear in the proof of (2.11) in Lemma 2.9.

LEMMA 2.3. Let Ψ_{2l-1} be defined for pairwise distinct $x_{-m}, \dots, x_n \in]-a, a[$ by

$$\Psi_{2l-1}(x_{-m}, \dots, x_n, x) = \frac{(4a)^{2l-1}}{(2l)!} \sum_{k=-m}^n \mu_k \left(B_{2l} \left(\frac{1}{2} + \frac{x+x_k}{4a} \right) + B_{2l} \left(\left\{ \frac{x-x_k}{4a} \right\} \right) \right)$$

where

$$\mu_k = \frac{\alpha_k}{\alpha_0} \quad \text{and} \quad \frac{1}{\alpha_k} = \prod_{\substack{-m \leq j \leq n \\ j \neq k}} \left(\sin \left(\pi \frac{x_k}{2a} \right) - \sin \left(\pi \frac{x_j}{2a} \right) \right) \quad \text{for } k = -m, \dots, n.$$

Then

- a) For $l \geq 1$ the functions $\Psi_{2l-1}(\cdot, \dots, \cdot, \pm a)$ have continuous extensions $\Psi_{2l-1}^*(\cdot, \dots, \cdot, \pm a)$ defined for $x_{-m}, \dots, x_n \in]-a, a[$.
- b) If $2l \geq m+n+2$ the function Ψ_{2l-1} has a continuous extension Ψ_{2l-1}^* defined for $x_{-m}, \dots, x_n \in]-a, a[$ and $x \in [-a, a]$.
- c) If $2r \geq m+n+2$ and $f \in C^{2r}[-a, a]$ is defined on $[-a, a]$ and vanishes at x_k with $k \neq 0$ where $-a < x_{-m} \leq \dots \leq x_{-1} < x_0 < x_1 \leq \dots \leq x_n < a$ and the x_k are numbered taking into account their multiplicity, then we have the identity

$$(2.6) \quad f(x_0) = \sum_{k=1}^r f^{(2k-1)}(a) \Psi_{2k-1}^*(a) - \sum_{k=1}^r f^{(2k-1)}(-a) \Psi_{2k-1}^*(-a) \\ - \int_{-a}^a f^{(2r)}(x) \Psi_{2r-1}^*(x) dx$$

where for short $\Psi_{2k-1}^*(\pm a)$ and $\Psi_{2r-1}^*(x)$ stand for $\Psi_{2k-1}^*(\cdot, \dots, \cdot, \pm a)$ and $\Psi_{2r-1}^*(\cdot, \dots, \cdot, x)$.

PROOF. Introducing the function h defined by

$$h(t, x) = \frac{(4a)^{2l-1}}{(2l)!} \left(B_{2l} \left(\frac{1}{2} + \frac{x}{4a} + \frac{1}{2\pi} \operatorname{Arcsin} t \right) + B_{2l} \left(\left\{ \frac{x}{4a} - \frac{1}{2\pi} \operatorname{Arcsin} t \right\} \right) \right)$$

we have

$$\Psi_{2l-1}(x_{-m}, \dots, x_n, \pm a) = \frac{1}{\alpha_0} \sum_{k=-m}^n \alpha_k h \left(\sin \left(\pi \frac{x_k}{2a} \right), \pm a \right)$$

for pairwise distinct $x_{-m}, \dots, x_n \in]-a, a[$ and assertion a) holds since the functions $h(\cdot, \pm a)$ belong to $C^\infty]-1, 1[$.

By definition the function h belongs to $C^{2l-2}]-1, 1[\times]-a, a[$ and the assertion b) is a consequence of the representation formula (2.5) since we have $2l-2 \geq m+n$. For pairwise distinct $x_{-m}, \dots, x_n \in]-a, a[$ the left-hand side of identity (2.1) reads

$$\frac{1}{\alpha_0} \sum_{k=-m}^n \alpha_k f \left(\frac{2a}{\pi} \operatorname{Arcsin} \left(\sin \left(\pi \frac{x_k}{2a} \right) \right) \right)$$

and thanks to Lemma 2.2 this expression, and hence the identity (2.1), extend to $x_{-m}, \dots, x_n \in]-a, a[$. One completes the proof of c) by observing, thanks to Lemma 2.2, that the left-hand side reduces to $f(x_0)$ when the x_k are zeros of multiplicity r_k of f . \square

The results stated in Lemma 2.4 play a central role in the proof of properties of functions $\Psi_{2l-1}^*(\cdot, \dots, \cdot, \pm a)$.

LEMMA 2.4. *For all $m, k \in \mathbb{N}^*$ we have the inequality*

$$(-1)^{m+1} \frac{d^k}{dx^k} B_{2m} \left(\frac{1}{2} + \frac{1}{\pi} \operatorname{Arcsin} \sqrt{x} \right) > 0 \quad \text{for } x \in [0, 1[.$$

The proof of Lemma 2.4 requires two technical results given in Lemmas 2.5 and 2.6.

LEMMA 2.5. *For all $k \in \mathbb{N}$ we have the Taylor expansion*

$$(\operatorname{Arcsin} x)^{2k} = \sum_{l=0}^{\infty} \frac{(2k)!}{(2l)!} 2^{2l-2k} b_{k,l} x^{2l} \quad \text{for } x \in [-1, 1]$$

where $b_{k,l}$ are integers defined recursively by

$$\begin{aligned} b_{0,0} &= 1 \text{ and } b_{k,0} = b_{0,l} = 0 \quad \text{for } k, l \geq 1 \\ b_{k+1,l+1} &= b_{k,l} + l^2 b_{k+1,l} \quad \text{for } k, l \geq 0. \end{aligned}$$

PROOF. We note first that the functions $f_{2k}(x) := (\text{Arcsin } x)^{2k}$ satisfy

$$(1-x^2)f_{2k+2}''(x) - x f_{2k+2}'(x) - (2k+2)(2k+1)f_{2k}(x) = 0 \quad \text{for } x \in]-1, 1[.$$

From the definition of f_{2k} and the above equality it follows that numbers $c_{k,l}$ defined by $f_{2k}(x) = \sum_{l=0}^{\infty} c_{k,l} x^{2l}$ for $x \in [-1, 1]$ are uniquely determined by the recurrence relations

$$\begin{aligned} c_{0,0} &= 1 \text{ and } c_{k,0} = c_{0,l} = 0 \quad \text{for } k, l \geq 1 \\ (2l+2)(2l+1)c_{k+1,l+1} - 4l^2 c_{k+1,l} - (2k+2)(2k+1)c_{k,l} &= 0 \quad \text{for } k, l \geq 0. \end{aligned}$$

A simple check shows that $c_{k,l} = \frac{(2k)!}{(2l)!} 2^{2l-2k} b_{k,l}$. □

LEMMA 2.6. *Let $b_{k,l}$ be the numbers defined in Lemma 2.5. Then*

$$(2.7) \quad \lim_{l \rightarrow \infty} \frac{b_{k,l}}{((l-1)!)^2} = \frac{\pi^{2k-2}}{(2k-1)!} \quad \text{for all } k \geq 1.$$

PROOF. From the definition of numbers $b_{k,l}$ we infer that $b_{1,l} = ((l-1)!)^2$ for $l \geq 1$. Thus relation (2.7) is trivially true for $k = 1$. We then assume $k \geq 2$. As $b_{j,1} = 0$ for $j \geq 2$ the numbers $d_{j,l}$ defined for $j, l \geq 1$ by $d_{j,l} = \frac{b_{j,l}}{((l-1)!)^2}$ satisfy the recurrence relations

$$\begin{aligned} d_{j,1} &= 0 \text{ and } d_{1,l} = 1 \quad \text{for } j \geq 2 \text{ and } l \geq 1, \\ d_{j+1,l+1} &= \frac{1}{l^2} d_{j,l} + d_{j+1,l} \quad \text{for } j, l \geq 1. \end{aligned}$$

Using the fact that $d_{j-1,l} = 0$ for $l = 1, \dots, j-2$ we get first for $j \geq 2$ the equality

$$d_{j,n_j} = \sum_{n_{j-1}=j-1}^{n_j-1} \frac{1}{n_{j-1}^2} d_{j-1,n_{j-1}}$$

which we iterate to obtain

$$d_{k,l} = \sum_{n_{k-1}=k-1}^{l-1} \frac{1}{n_{k-1}^2} \sum_{n_{k-2}=k-2}^{n_{k-1}-1} \frac{1}{n_{k-2}^2} \cdots \sum_{n_2=2}^{n_3-1} \frac{1}{n_2^2} \sum_{n_1=1}^{n_2-1} \frac{1}{n_1^2}.$$

This leads to

$$\lim_{l \rightarrow \infty} d_{k,l} = \sum_{n_{k-1} > n_{k-2} > \cdots > n_2 > n_1 > 0} \prod_{j=1}^{k-1} \frac{1}{n_j^2}$$

and we recognize in the right-hand side the number $\zeta(\{2\}_{(k-1)})$ whose value, given in [2], is equal to the right-hand side of (2.7). □

PROOF OF LEMMA 2.4. It suffices to prove that the numbers $e_{m,l}$ defined by

$$(2.8) \quad (-1)^{m+1} B_{2m} \left(\frac{1}{2} + \frac{1}{\pi} \operatorname{Arcsin} x \right) = \sum_{l=0}^{\infty} e_{m,l} x^{2l}$$

satisfy $e_{m,l} > 0$ for all $m, l \in \mathbb{N}^*$. Using Taylor's formula and the evenness of function $B_{2m}(\frac{1}{2} + \frac{t}{\pi})$, we have

$$B_{2m} \left(\frac{1}{2} + \frac{t}{\pi} \right) = \sum_{k=0}^m \frac{1}{(2k)!} B_{2m}^{(2k)} \left(\frac{1}{2} \right) \left(\frac{t}{\pi} \right)^{2k} = \sum_{k=0}^m \binom{2m}{2k} B_{2m-2k} \left(\frac{1}{2} \right) \left(\frac{t}{\pi} \right)^{2k}$$

and the Taylor expansion of $(\operatorname{Arcsin} x)^{2k}$ given in Lemma 2.5 leads to

$$\begin{aligned} B_{2m} \left(\frac{1}{2} + \frac{1}{\pi} \operatorname{Arcsin} x \right) &= \sum_{k=0}^m \left(\binom{2m}{2k} B_{2m-2k} \left(\frac{1}{2} \right) \pi^{-2k} \sum_{l=0}^{\infty} \frac{(2k)!}{(2l)!} 2^{2l-2k} b_{k,l} x^{2l} \right) \\ &= \frac{(2m)!}{(2\pi)^{2m}} \sum_{k=0}^m \left(\frac{(2\pi)^{2m-2k}}{(2m-2k)!} B_{2m-2k} \left(\frac{1}{2} \right) \sum_{l=0}^{\infty} \frac{2^{2l}}{(2l)!} b_{k,l} x^{2l} \right). \end{aligned}$$

We then change the order of summation to get

$$(2.9) \quad (-1)^{m+1} B_{2m} \left(\frac{1}{2} + \frac{1}{\pi} \operatorname{Arcsin} x \right) = \frac{(2m)!}{(2\pi)^{2m}} \sum_{l=0}^{\infty} \frac{2^{2l}}{(2l)!} f_{m,l} x^{2l}$$

where

$$f_{m,l} = (-1)^{m+1} \sum_{k=0}^m \frac{(2\pi)^{2m-2k}}{(2m-2k)!} B_{2m-2k} \left(\frac{1}{2} \right) b_{k,l}.$$

We prove by recurrence over m that $f_{m,l} > 0$ for $m, l \geq 1$. To this end we set $g_{m,l} = \frac{f_{m,l}}{((l-1)!)^2}$ for $m, l \geq 1$ and since $b_{0,l} = 0$ for $l \geq 1$ we have

$$\begin{aligned} g_{m+1,l+1} &= \frac{(-1)^{m+2}}{(l!)^2} \sum_{k=1}^{m+1} \frac{(2\pi)^{2m+2-2k}}{(2m+2-2k)!} B_{2m+2-2k} \left(\frac{1}{2} \right) b_{k,l+1} \\ &= \frac{(-1)^{m+2}}{(l!)^2} \sum_{k=1}^{m+1} \frac{(2\pi)^{2m+2-2k}}{(2m+2-2k)!} B_{2m+2-2k} \left(\frac{1}{2} \right) (b_{k-1,l} + l^2 b_{k,l}) \\ &= \frac{(-1)^{m+2}}{(l!)^2} \sum_{k=1}^{m+1} \frac{(2\pi)^{2m+2-2k}}{(2m+2-2k)!} B_{2m+2-2k} \left(\frac{1}{2} \right) b_{k-1,l} + g_{m+1,l} \\ &= -\frac{(-1)^{m+1}}{(l!)^2} \sum_{k=0}^m \frac{(2\pi)^{2m-2k}}{(2m-2k)!} B_{2m-2k} \left(\frac{1}{2} \right) b_{k,l} + g_{m+1,l} \\ &= -\frac{1}{l^2} g_{m,l} + g_{m+1,l} \end{aligned}$$

and this implies that

$$g_{m+1,l+1} + \frac{1}{l^2} g_{m,l} = g_{m+1,l} \quad \text{for } l \geq 1.$$

We have $g_{1,l} = f_{1,l} = 1$ for all $l \geq 1$. Let us suppose that $g_{m,l} > 0$ for all $l \geq 1$. Then $g_{m+1,l+1} < g_{m+1,l}$ and it follows that $g_{m+1,l} > \lim_{l \rightarrow \infty} g_{m+1,l}$. Thanks to Lemma 2.6 we have

$$(2.10) \quad \lim_{l \rightarrow \infty} g_{m+1,l} = (-1)^{m+2} \sum_{k=1}^{m+1} \frac{(2\pi)^{2m+2-2k}}{(2m+2-2k)!} B_{2m+2-2k} \left(\frac{1}{2} \right) \frac{\pi^{2k-2}}{(2k-1)!}$$

$$= (-1)^{m+2} \pi^{2m} \sum_{k=1}^{m+1} \frac{2^{2m+2-2k}}{(2m+2-2k)!(2k-1)!} B_{2m+2-2k} \left(\frac{1}{2} \right)$$

and using $B_j(\frac{1}{2}) = 0$ for all odd j and the formula

$$B_n(x+y) = \sum_{j=0}^n \binom{n}{j} B_j(x) y^{n-j}$$

we check that the sum which appears in (2.10) is equal to

$$\sum_{j=0}^{2m+1} \frac{2^j}{j!(2m+1-j)!} B_j \left(\frac{1}{2} \right) = \frac{2^{2m+1}}{(2m+1)!} \sum_{j=0}^{2m+1} \binom{2m+1}{j} B_j \left(\frac{1}{2} \right) \left(\frac{1}{2} \right)^{2m+1-j}$$

$$= \frac{2^{2m+1}}{(2m+1)!} B_{2m+1}(1) = 0.$$

Hence $g_{m,l} > 0$ for $m, l \geq 1$ and this implies, thanks to (2.9), that the numbers $e_{m,l}$ defined by (2.8) are positive for $m, l \geq 1$. \square

We are now in position to prove main properties of functions $\Psi_{2l-1}^*(\cdot, \dots, \cdot, \pm a)$.

LEMMA 2.7. *Let $\Psi_{2l-1}^*(\cdot, \dots, \cdot, \pm a)$ be the functions defined in Lemma 2.3. Then*

- a) $(-1)^{n+l+1} \Psi_{2l-1}^*(x_{-m}, \dots, x_n, a) > 0$ for $x_{-m}, \dots, x_n \in]-a, a[$.
- b) $(-1)^{m+l+1} \Psi_{2l-1}^*(x_{-m}, \dots, x_n, -a) > 0$ for $x_{-m}, \dots, x_n \in]-a, a[$.

PROOF. For pairwise distinct $x_{-m}, \dots, x_n \in]-a, a[$ we have

$$\Psi_{2l-1}^*(x_{-m}, \dots, x_n, \pm a) = 2 \frac{(4a)^{2l-1}}{(2l)!} \sum_{k=-m}^n \mu_k B_{2l} \left(\frac{1}{2} + \frac{\pm a + x_k}{4a} \right)$$

since the function $B_{2m}(\frac{1}{2} + t)$ is even and then

$$(-1)^{n+l+1} \Psi_{2l-1}^*(x_{-m}, \dots, x_n, \pm a)$$

$$= 2 \frac{(4a)^{2l-1}}{(2l)!} \left(\frac{(-1)^n}{\alpha_0} \right) \sum_{k=-m}^n \alpha_k (-1)^{l+1} B_{2l} \left(\frac{1}{2} + \frac{\pm a + x_k}{4a} \right).$$

The first two terms of the right-hand side are positive and the third term reads $\sum_{k=-m}^n \alpha_k h_{\pm}(\sin(\pi \frac{x_k}{2a}))$ where

$$h_{\pm}(t) = (-1)^{l+1} B_{2l} \left(\frac{3}{4} \pm \frac{1}{2\pi} \operatorname{Arcsin} t \right) \quad \text{for } t \in [-1, 1].$$

The identities

$$\frac{3}{4} \pm \frac{1}{2\pi} \operatorname{Arcsin} t = \frac{1}{2} + \frac{1}{\pi} \operatorname{Arcsin} \sqrt{\frac{1 \pm t}{2}} \quad \text{for } t \in [-1, 1]$$

together with Lemma 2.4 show that $h_+^{(m+n)}$ and $(-1)^{m+n} h_-^{(m+n)}$ are positive on $] -1, 1[$ and the conclusion holds by Lemma 2.2. \square

The last point is to bound the integral which appears in the right-hand side of the identity (2.6). This is the content of Lemma 2.9, whose proof needs the following result.

LEMMA 2.8. *Let $b_{r,s}$ the numbers defined for integers $r \geq 4$ and $s \geq 0$ by*

$$b_{r,s} = \left(\frac{r}{r+s} \right)^{2r \log r - 1} \binom{2r+s-1}{s}.$$

Then $\sum_{s=0}^{\infty} b_{r,s}^2 = 1 + o(1)$ as $r \rightarrow \infty$.

PROOF. We have $b_{r,s} = O_r(s^{-2r \log r + 2r}) = O_r(s^{-1})$ and hence $\sum_{s=0}^{\infty} b_{r,s}^2$ is convergent. We now prove that $b_{r,s} \leq b_{4,s}$ for $r \geq 4$. We have $\log b_{r,s} = g(r, s)$ where the function g is defined for $(x, y) \in [4, \infty[\times [0, \infty[$ by

$$g(x, y) = (2x \log x - 1) \log \left(\frac{x}{x+y} \right) + \log \Gamma(2x+y) - \log \Gamma(y+1) - \log \Gamma(2x).$$

Straightforward computations lead to

$$\frac{\partial g}{\partial x}(x, y) = (2 \log x + 2) \log \left(\frac{x}{x+y} \right) + (2x \log x - 1) \frac{y}{x(x+y)} + 2\Psi(2x+y) - 2\Psi(2x)$$

and

$$\frac{\partial^2 g}{\partial y \partial x}(x, y) = -\frac{1+2x+2y+2y \log x}{(x+y)^2} + 2\Psi'(2x+y)$$

where Ψ is the derivative of $\log \Gamma$. We have $\frac{\partial g}{\partial x}(x, 0) = 0$ and moreover since $\Psi'(z) = \sum_{k=0}^{\infty} \frac{1}{(z+k)^2}$, we get $\Psi'(z) \leq \frac{1}{z} + \frac{1}{z^2}$ for $z > 0$ and therefore

$$\begin{aligned} \frac{\partial^2 g}{\partial y \partial x}(x, y) &\leq -\frac{1+2x+2y+2y \log x}{(x+y)^2} + \frac{2}{2x+y} + \frac{2}{(2x+y)^2} \\ &= -\left(\frac{4x^3 + 2(1+3y)x^2 + (2x-1)y^2}{(x+y)^2(2x+y)^2} + \frac{2y \log x}{(x+y)^2} \right) \leq 0. \end{aligned}$$

Hence $\frac{\partial g}{\partial x}(x, y) \leq \frac{\partial g}{\partial x}(x, 0) = 0$ and this implies that $g(x, y) \leq g(4, y)$ and hence $b_{r,s} \leq b_{4,s}$ for $r \geq 4$.

Let $\epsilon > 0$ and s_0 be such that $\sum_{s=s_0}^{\infty} b_{4,s}^2 \leq \frac{\epsilon}{2}$. Since $b_{r,s} \rightarrow 0$ as $r \rightarrow \infty$ for $s \geq 1$, there exists r_0 such that $\sum_{s=1}^{s_0-1} b_{r,s}^2 \leq \frac{\epsilon}{2}$ for $r \geq r_0$. Hence

$$1 \leq \sum_{s=0}^{\infty} b_{r,s}^2 \leq b_{r,0}^2 + \sum_{s=1}^{s_0-1} b_{r,s}^2 + \sum_{s=s_0}^{\infty} b_{4,s}^2 \leq 1 + \epsilon \quad \text{for } r \geq r_0.$$

The proof is complete. \square

LEMMA 2.9. *For positive integers m, n and l which satisfy $m + n \geq 4$ and $l \geq (m + n) \log(m + n)$ and for $x_{-m}, \dots, x_n \in]-a, a[$, let Ψ_{2l-1}^* be the function defined in Lemma 2.3. Then*

$$\|\Psi_{2l-1}^*\|_2 = \frac{2^{m+n-1}}{|\alpha_0| \sqrt{a}} \left(\frac{2a}{(m+n)\pi} \right)^{2l} (1 + o(1)) \quad \text{as } m+n \rightarrow \infty$$

where

$$\|\Psi_{2l-1}^*\|_2^2 = \int_{-a}^a (\Psi_{2l-1}^*(x_{-m}, \dots, x_n, x))^2 dx.$$

PROOF. The use of the Fourier series expansion

$$B_{2l}(x) = (-1)^{l+1} 2((2l)!) \sum_{j=1}^{\infty} \frac{1}{(2j\pi)^{2l}} \cos(2j\pi x) \quad \text{for } x \in [0, 1]$$

and the identity $\cos \alpha + \cos \beta = 2 \cos(\frac{\alpha+\beta}{2}) \cos(\frac{\alpha-\beta}{2})$ lead, for pairwise distinct $x_{-m}, \dots, x_n \in]-a, a[$, to the expression

$$\begin{aligned} & \Psi_{2l-1}^*(x_{-m}, \dots, x_n, x) \\ &= (-1)^{l+1} 2 \frac{(2a)^{2l-1}}{\alpha_0 \pi^{2l}} \sum_{j=1}^{\infty} \frac{1}{j^{2l}} \left(\sum_{k=-m}^n \alpha_k \cos \left(j\pi \left(\frac{1}{2} + \frac{x_k}{2a} \right) \right) \right) \cos \left(j\pi \left(\frac{1}{2} + \frac{x}{2a} \right) \right). \end{aligned}$$

Using the identity $\cos(j\pi(\frac{1}{2} + y)) = (-1)^j T_j(\sin(\pi y))$ and introducing the numbers $a_{j,k} = (-1)^j T_j(\sin(\pi \frac{x_k}{2a}))$ we have $\sum_{k=-m}^n \alpha_k a_{j,k} = 0$ for $j = 1, \dots, m+n-1$, this is crucial, and, therefore

$$\begin{aligned} (2.11) \quad & \Psi_{2l-1}^*(x_{-m}, \dots, x_n, x) \\ &= (-1)^{l+1} 2 \frac{(2a)^{2l-1}}{\alpha_0 \pi^{2l}} \sum_{j=m+n}^{\infty} \frac{1}{j^{2l}} \left(\sum_{k=-m}^n \alpha_k a_{j,k} \right) \cos \left(j\pi \left(\frac{1}{2} + \frac{x}{2a} \right) \right). \end{aligned}$$

Using Lemma 2.2, squaring (2.11) and integrating on the interval $[-a, a]$, we get

$$(2.12) \quad \|\Psi_{2l-1}^*\|_2^2 = \left(2 \frac{(2a)^{2l-1}}{\alpha_0 \pi^{2l}} \right)^2 \sum_{j=m+n}^{\infty} \frac{1}{j^{4l}} \left(\frac{T_j^{(m+n)}(\tau_j)}{(m+n)!} \right)^2 a$$

for some $\tau_j \in]-1, 1[$. It is well known [10] that

$$\max_{-1 \leq x \leq 1} |T_j^{(m+n)}(x)| = T_j^{(m+n)}(1) = 2^{m+n-1} (m+n-1)! j \binom{m+n+j-1}{j-m-n}$$

for $j = m + n, m + n + 1, \dots$ and then

$$\|\Psi_{2l-1}^*\|_2^2 \leq \left(\frac{2^{m+n-1}}{\alpha_0 \sqrt{a}} \left(\frac{2a}{(m+n)\pi} \right)^{2l} \right)^2 \sum_{j=m+n}^{\infty} \left(\frac{m+n}{j} \right)^{4l-2} \binom{m+n+j-1}{j-m-n}^2.$$

We set $j = m + n + s$ and since $l \geq (m+n) \log(m+n)$ we have

$$\begin{aligned} \|\Psi_{2l-1}^*\|_2^2 &\leq \left(\frac{2^{m+n-1}}{\alpha_0 \sqrt{a}} \left(\frac{2a}{(m+n)\pi} \right)^{2l} \right)^2 \sum_{s=0}^{\infty} \left(\left(\frac{m+n}{m+n+s} \right)^{2l-1} \binom{2(m+n)+s-1}{s} \right)^2 \\ &\leq \left(\frac{2^{m+n-1}}{\alpha_0 \sqrt{a}} \left(\frac{2a}{(m+n)\pi} \right)^{2l} \right)^2 \sum_{s=0}^{\infty} b_{m+n,s}^2 \\ &= \left(\frac{2^{m+n-1}}{\alpha_0 \sqrt{a}} \left(\frac{2a}{(m+n)\pi} \right)^{2l} \right)^2 (1 + o(1)) \end{aligned}$$

as $m+n \rightarrow \infty$, thanks to Lemma 2.8. To complete the proof we compute a lower bound for $\|\Psi_{2l-1}^*\|_2^2$ using the first term in the series which appears in the right-hand side of (2.12). \square

3. Proof of Theorem

In this section we assume that the Riemann hypothesis is true. Our result is a consequence of identity (2.6) which for $Z(T)$ gives

$$(3.1) \quad Z(T) = \sum_{k=1}^K Z^{(2k-1)}(T+a) \Psi_{2k-1}^*(a) - \sum_{k=1}^K Z^{(2k-1)}(T-a) \Psi_{2k-1}^*(-a) - \int_{-a}^a Z^{(2K)}(T+x) \Psi_{2K-1}^*(x) dx$$

where $2K \geq m+n+2$ and for short $\Psi_{2k-1}^*(\pm a)$ and $\Psi_{2m-1}^*(x)$ stand respectively for $\Psi_{2k-1}^*(x_{-m}, \dots, x_n, \pm a)$ and $\Psi_{2m-1}^*(x_{-m}, \dots, x_n, x)$ and $x_k = \gamma_k - T$. The main step in the proof is to bound the integral which appears in the right-hand side of (3.1). In our proof we use the bound

$$(3.2) \quad \left| \int_T^{T+h} S(u) du \right| \leq \frac{\pi}{16} \frac{\log T}{(\log \log T)^2} + O\left(\frac{\log T \log \log \log T}{(\log \log T)^3} \right)$$

for $0 < h \leq \sqrt{T}$, due to Carneiro et al. [3].

LEMMA 3.1. *Let T be sufficiently large, $4(\log \log T)^{-1} \leq a \leq \sqrt{T}$, m and n be the integers defined in the introduction and let α_0 be the coefficient defined in Lemma 2.3 where $x_0 = 0$ and $x_k = \gamma_k - T$. Then*

$$(3.3) \quad -\log |\alpha_0| \leq -\frac{2a}{\pi} \theta'(T) \log 2 + \frac{1}{\sqrt{2}} \frac{\log T}{\log \log T} \left(1 + O\left(\frac{\log \log \log T}{\log \log T} \right) \right).$$

Further, if $S(T+a) - S(T-a) \geq 1$, then

$$(3.4) \quad m+n = \frac{2a}{\pi} \theta'(T) + \Delta S \left(1 + O\left(\frac{a^3}{T^2}\right) \right)$$

where $\Delta S = S(T+a) - S(T-a)$.

PROOF. By definition

$$\left| \frac{1}{\alpha_0} \right| = \prod_{\substack{-m \leq j \leq n \\ j \neq 0}} \left| \sin \left(\pi \frac{x_j}{2a} \right) \right|$$

and using (1.1) and Stieltjes integral we have

$$-\log |\alpha_0| = \int_{T-a}^{T+a} \log \left| \sin \left(\pi \frac{t-T}{2a} \right) \right| d \left(\frac{1}{\pi} \theta(t) + 1 + S(t) \right)$$

and an integration by parts leads to

$$\begin{aligned} -\log |\alpha_0| &= \frac{1}{\pi} \int_{T-a}^{T+a} \theta'(t) \log \left| \sin \left(\pi \frac{t-T}{2a} \right) \right| dt \\ &\quad - \frac{\pi}{2a} \int_{T-a}^{T+a} \cot \left(\pi \frac{t-T}{2a} \right) (S(t) - S(T)) dt. \end{aligned}$$

Now for $t \in [T-a, T+a]$ we have

$$\theta'(t) = \theta'(T) + \theta''(T)(t-T) + \frac{1}{2} \theta'''(\tau)(t-T)^2 \quad \text{for some } \tau \in [T-a, T+a]$$

and using $\theta'''(t) = O(1/t^2)$ together with

$$\int_{T-a}^{T+a} \log \left| \sin \left(\pi \frac{t-T}{2a} \right) \right| dt = \frac{4a}{\pi} \int_0^{\pi/2} \log \sin \tau d\tau = -2a \log 2$$

we get

$$(3.5) \quad \begin{aligned} -\log |\alpha_0| &= -\frac{2a}{\pi} \theta'(T) \log 2 \\ &\quad - \frac{\pi}{2a} \int_{T-a}^{T+a} \cot \left(\pi \frac{t-T}{2a} \right) (S(t) - S(T)) dt + O\left(\frac{a^3}{T^2}\right). \end{aligned}$$

Further, for $\eta \in]0, a]$ we have

$$\begin{aligned} &\int_{T-a}^{T+a} \cot \left(\pi \frac{t-T}{2a} \right) (S(t) - S(T)) dt \\ &= \int_{T-a}^{T-\eta} \cot \left(\pi \frac{t-T}{2a} \right) S(t) dt + \int_{T-\eta}^T \cot \left(\pi \frac{t-T}{2a} \right) (S(t) - S(T)) dt \\ &\quad + \int_T^{T+\eta} \cot \left(\pi \frac{t-T}{2a} \right) (S(t) - S(T)) dt + \int_{T+\eta}^{T+a} \cot \left(\pi \frac{t-T}{2a} \right) S(t) dt \end{aligned}$$

and we bound from below the third term using

$$S(t) - S(T) \geq -\frac{1}{\pi}(\theta(t) - \theta(T)) \geq -\frac{1}{\pi}\theta'(T+a)(t-T) \geq -\frac{1}{2\pi}\log T(t-T)$$

which, together with the inequality $\cot x < \frac{1}{x}$ for $x \in]0, \frac{\pi}{2}]$, implies

$$\int_T^{T+\eta} \cot\left(\pi\frac{t-T}{2a}\right)(S(t) - S(T)) dt \geq -\frac{\eta a}{\pi^2}\log T.$$

Applied to the fourth term, the second mean-value theorem gives

$$\begin{aligned} \int_{T+\eta}^{T+a} \cot\left(\pi\frac{t-T}{2a}\right)S(t) dt &= \cot\left(\pi\frac{\eta}{2a}\right) \int_{T+\eta}^{\tau} S(t) dt \\ &\geq -\frac{2a}{\pi\eta} \left(\frac{\pi}{16} \frac{\log T}{(\log \log T)^2} + O\left(\frac{\log T \log \log \log T}{(\log \log T)^3}\right) \right) \end{aligned}$$

where $\tau \in [T+\eta, T+a]$ and (3.2) has been used. Proceeding in the same way to bound from below the first and second term we finally get

$$\begin{aligned} -\frac{\pi}{2a} \int_{T-a}^{T+a} \cot\left(\pi\frac{t-T}{2a}\right)(S(t) - S(T)) dt \\ \leq \frac{\eta}{\pi} \log T + \frac{1}{\eta} \left(\frac{\pi}{8} \frac{\log T}{(\log \log T)^2} + O\left(\frac{\log T \log \log \log T}{(\log \log T)^3}\right) \right). \end{aligned}$$

We choose $\eta = \pi(2\sqrt{2}\log \log T)^{-1}$ and we complete the proof of (3.3) using (3.5). Since $m+n = N(T+a) - N(T-a)$, the Taylor formula leads to

$$m+n = \frac{1}{\pi}(\theta(T+a) - \theta(T-a)) + \Delta S = \frac{2a}{\pi}\theta'(T) + \Delta S \left(1 + O\left(\frac{a^3}{T^2}\right)\right).$$

which proves (3.4). \square

PROOF OF THEOREM 1.1. Either $K \leq K^* \log K^*$ and there is nothing to prove or $K > K^* \log K^*$ and this implies that $K \geq (m+n)\log(m+n)$. Assuming T large enough we have also $m+n \geq 4$. By definition of K and thanks to Lemma 2.7, the first sum in the right-hand side of identity (3.1) is nonpositive and the second one is nonnegative. Using Cauchy-Schwarz inequality, (1.4) and Lemma 2.9, we get

$$Z(T) \leq \|Z^{(2K)}\|_2 \|\Psi_{2K-1}^*\|_2 \leq C_{2K} Z(T)$$

where

$$\begin{aligned} C_{2K} &= c_{2K} 2^{m+n-\frac{1}{2}} |\alpha_0|^{-1} \left(\frac{2a\theta'(T)}{(m+n)\pi} \right)^{2K} (1+o(1)) \\ &\leq c_{2K} 2^{m+n} |\alpha_0|^{-1} \left(1 + \pi \frac{\Delta S}{2a\theta'(T)} \left(1 + O\left(\frac{a^3}{T^2}\right) \right) \right)^{-2K} \end{aligned}$$

for T sufficiently large. Since $C_{2K} \geq 1$ we have $\log C_{2K} \geq 0$ and this implies that

$$\log c_{2K} + (m+n) \log 2 - \log |\alpha_0| - 2K \log \left(1 + \pi \frac{\Delta S}{2a\theta'(T)} \left(1 + O\left(\frac{a^3}{T^2}\right) \right) \right) \geq 0.$$

Finally, as $\Delta S = O(\log T (\log \log T)^{-1})$ and thanks to Lemma 3.1, we get

$$\begin{aligned} 2K\pi \frac{\Delta S}{2a\theta'(T)} \left(1 + O\left(\frac{1}{\log \log T}\right) \right) \\ \leq \left(\log c_{2K} + \Delta S \log 2 + \frac{1}{\sqrt{2}} \frac{\log T}{\log \log T} \right) \left(1 + O\left(\frac{\log \log \log T}{\log \log T}\right) \right) \end{aligned}$$

and the proof is complete. \square

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