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# Analytical Expressions for Lightning Electromagnetic Fields with Arbitrary Channel Base Current. Part I: Theory

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*Abstract*— **The paper provides analytical expressions for the electromagnetic fields generated by a lightning return stroke characterized by a channel base current with arbitrary time waveform, in presence of either a perfectly conducting or a lossy ground, assuming the transmission line model for the current along the channel. In this second case, a time domain analytical expression for the Cooray-Rubinstein formula is presented. The main idea that leads to the derivation of analytical formulas consists of dividing the channel into intervals in which the distance between the field source point and the observation point can be approximated with a linear function of the time and of the spatial coordinates of both points. In the companion paper, a detailed comparison is proposed with the classical (numerical) approach highlighting excellent agreement both at close and far distances, considering all the values of practical interest for the ground conductivity. Moreover, the method guarantees a meaningful improvement in the computational performance.**

*Index Terms*— **Lightning electromagnetic fields, Channel base current, Engineering models**

## I. Introduction

INDIRECT lightning strikes generate overvoltages that are among the most frequent causes of damages or interruptions in distribution systems. Since their computation can be complex and very time consuming, having at one's disposal reliable tools that perform fast and accurate indirect overvoltage calculations is therefore of great importance while designing the lightning protection system. Four models based on the transmission line approximation have been successfully validated, shown to be complete, and shown to be equivalent to each other in the literature: the Taylor et al. model [1], the Agrawal et al. [2], the Rachidi model [3], and the modified Rusck model [4, 5][1]. Different methods have been developed to evaluate lightning-induced overvoltages; such methods can be essentially divided into two broad categories: analytical [11-15] and numerical [16-20]. All of the methods require as inputs specific components of the lightning electromagnetic fields that have to be computed in a fast but accurate way. To do this, a first possibility is to assume that the ground is a perfect conductor and the lightning channel is vertical. Under those conditions, expressions for the electromagnetic fields involving an integral over the channel have been derived both in the frequency and in the time domain [21]. However, it has been shown [21] that the finite conductivity of the ground plays an important role, especially in the evaluation of the radial component of the electric field that represents the source term of the coupling equations in the model of Agrawal et al. The exact expressions of the field in the presence of a lossy ground involve the so-called Sommerfeld integrals, whose exact evaluation, performed in [22-24], is prohibitive from a computational point of view. For this reason, an approximate approach has been proposed by Cooray and Rubinstein in [25, 26] in the frequency domain and then some alternative expressions in the time domain have been presented in [27-30]. However, to evaluate the overvoltages induced by a nearby lightning strike, it is necessary to calculate the radial component of the electric field at the points in which the line is discretized, and the vertical component of the electric field at the line extremities [2]. Moreover, when one aims at evaluating the lightning performance of a distribution system, a statistical analysis has to be conducted, for which thousands of lightning events have to be randomly generated at different points of impact and having different channel base current parameters [19, 20, 31, 32]. It is then apparent that a huge number of field calculations is required, which makes the computational performance of such calculations a crucial aspect for this kind of application. The attempt of providing analytical formulas for lightning electromagnetic fields is not new. First attempts were presented in [4, 33-35]. However, these studies are limited to some specific channel-base current expressions: while [4, 34] considered only a step-function, the authors of [35] provided a survey on many different approximations for the radial electric field in the presence of a lossy ground, where their accuracy was tested against exact expressions derived in the case of a double-exponential channel-base current waveform. More recently, two main solutions have been adopted in the literature: i) analytical formulas for the electromagnetic fields over a perfectly conducting ground have been derived in [36] assuming that the channel base current is trapezoidal, and ii) a field waveform database [20] containing electromagnetic fields generated from a current with a specified time domain waveform with unitary peak at different distances. The field waveforms required as inputs to the coupling equations are then calculated interpolating the database elements. According to

[1] Two early models, the Rusck model [6], [7] and the Chowduri-Gross model [8]. exist but they have been shown to be incomplete [9] [10].

[37], the first approach *"is somewhat limited when the presence of surge arresters and flashovers occurrence is taken into account"*. On the other hand, the database approach exploits the linear relationship between current peak and electromagnetic fields. However, when one aims at accounting for the front time effect, the computational convenience of the method fails because one database should be constructed for any considered value of the front time. To overcome these problems, this paper proposes analytical formulas for lightning electromagnetic fields generated by an arbitrarily-shaped channel base current over either a perfectly conducting or a lossy ground. The idea of the method consists of dividing the channel into intervals in which the distance between the source and the observation field points can be approximated with a linear function of the time and of their spatial coordinates.

The paper is organized as follows. In Section II, we present the general idea that supports the analytical derivation. This section is aimed at easing the understanding of the analytical derivations presented in Sections III and IV. Section III is dedicated to the derivation of the fields over an ideal ground. Section IV proposes the time domain analytical expression for the Cooray Rubinstein formula. Section V proposes a summary of the procedure and finally, Section VI is dedicated to conclusions. A complete validation of the method is presented in the companion paper.

## II. The General Idea of the Proposed Approach

The aim of this section is to present the general idea behind the method, making use of a graphical sketch. mathematical details will be provided in the following sections.

Let us consider the situation depicted in Fig. 1 representing the lightning channel (H being its height). Assuming a perfectly-conducting ground and assuming that the lightning current starts propagating from the channel base ($z$'=0) at $t$=0, and applying the method of images, the azimuthal component of the magnetic field produced at the observation point $Q(r,\phi,z)$ is given by [21]:

$$H_\phi^{id}(r,z,t)=\frac{r}{4\pi}\int_{-H}^{H}\left[\frac{1}{R^3(z')}i_0\left(\xi(z',t)\right)+\right.$$
$$\left.+\frac{1}{c_0R^2(z')}i'_0\left(\xi(z',t)\right)\right]\cdot P(|z'|)1\left(\xi(z')\right)dz' \tag{1}$$

where

$$\xi(z',t)=t-\frac{R(z')}{c_0}-\frac{|z'|}{v} \tag{2}$$

In (1) and (2), $c_0$ and $v$ are the speed of light in vacuum and the return stroke speed, respectively, $P$ is the attenuation function [38], which can be specified for any given engineering return stroke model, $1(t)$ is the Heaviside function, $i_0(t)$ is the channel base current (assumed to be zero for negative time), and $i'_0$ is its first time derivative. $R$ is the distance between the source and the observation point, defined as

$$R(z')=\sqrt{r^2+(z-z')^2} \tag{3}$$

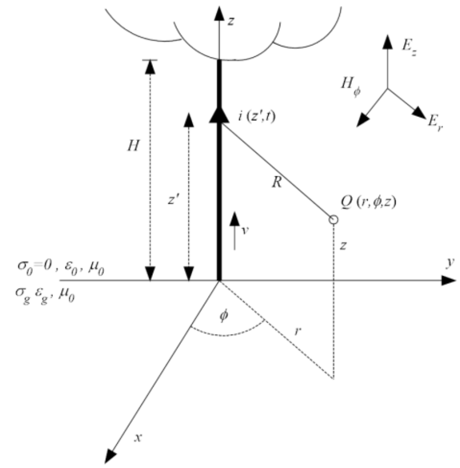


Fig. 1 Geometry of the problem

It can be shown that, after some algebraic manipulations [39] and assuming P($z$')=1 (Transmission Line Model), the magnetic field can be written as :

$$H_\phi^{id}(r,z,t)=\frac{c_0}{2\pi rv}i_0\left(t-\frac{R(0)}{c_0}\right)+$$
$$+\frac{c_0}{4\pi r}\left(\frac{1}{c_0^2}-\frac{1}{v^2}\right)\int_{-H}^{H}i'_0\left(\xi(z',t)\right)1\left(\xi(z',t)\right)dz' \tag{4}$$

The Heaviside function in (4) defines the portion of the channel that contributes to the fields at the observation point $Q$ at time $t$, given by solving for $z$' in the inequality $\xi(z',t)\geq 0$ .

As a consequence, indicating with $t_*$ the time required by the fields produced by the dipole located at the channel base to reach $Q$, the magnetic field is zero at that point for $t<t_*$, as shown in Fig. *2* (a), where the left view is the channel representation (the yellow line representing the propagation of the current along the channel), while the right view represents the time domain waveform of the magnetic field at $Q$.

Now, let us divide the channel into segments. Using $t_1$ to designate the time at which the fields produced by the last dipole in the first segment reach point $Q$, then for $t_*<t<t_1$, a portion of the first segment (the complete segment is indicated by the red curly bracket) contributes to the field at $Q$. If one approximates $\xi(z',t)$ as a linear function of $z$', then: i) the integral appearing in (4) can be solved analytically by substitution, and ii) inequality $\xi(z',t)\geq 0$ can be solved analytically allowing the determination of the actual integration domain as a function of $t$ (Fig. 2(b)).

Repeating the same considerations for the second segment ($t_2$ being the time necessary for the fields of the dipole located at the upper end of the second segment to propagate to the observation point), one can approximate $\xi(z',t)$ with another linear function of $z$'. This allows the analytical evaluation of the magnetic field at $Q$ for $t1<t<t2$, in which the complete first segment and part of the second one contribute to the field (Fig. 2 (c)).

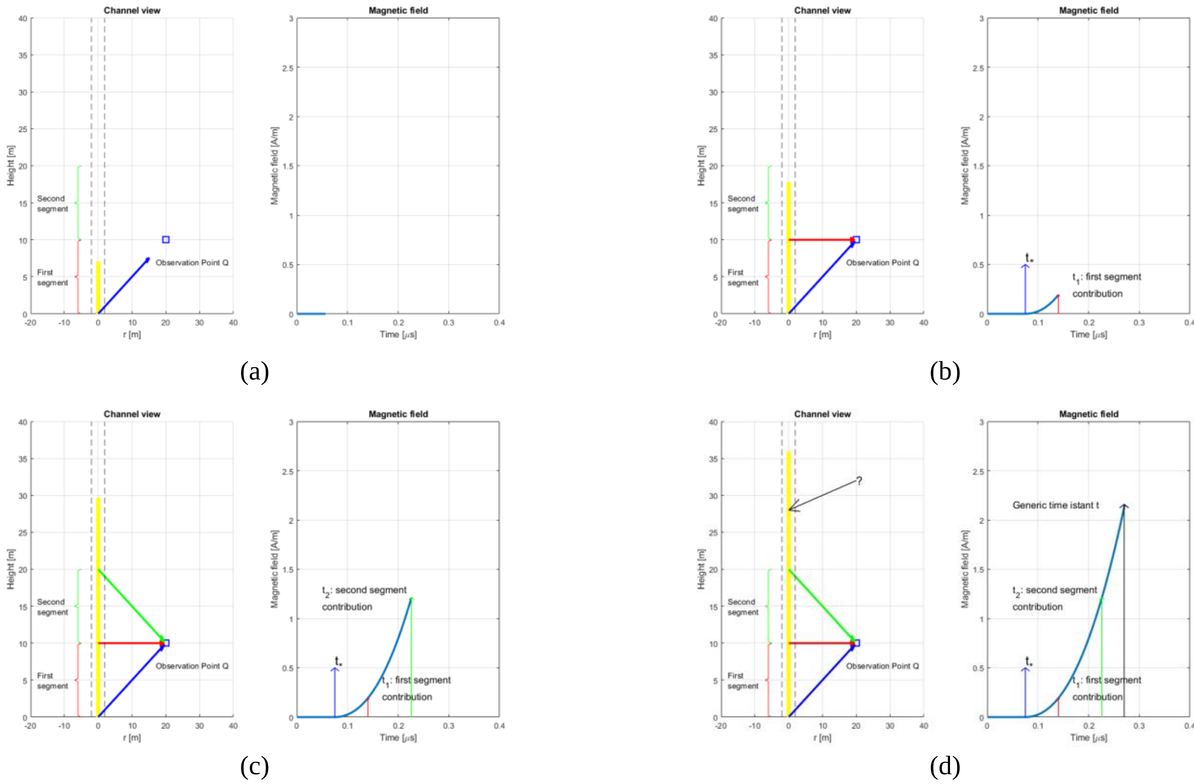


Fig. 2 Graphical sketches illustrating the time intervals at which the different channel segments contribute to the field at observation point $Q$: $t<t_*$ in panel (a); $t_*<t<t_1$ panel (b); $t_1<t<t_2$ panel (c); generic $t> t_2$ panel (d)

Repeating these steps, one can define the following time sequence $t_*<t_1<t_2<\ldots<t_n$; as a consequence, comparing the generic time instant $t$ with the elements of the sequence, one can find the last segment that (fully or partly) contributes to the field at $Q$. Obtaining the linear approximation of $\xi(z',t)$ for each segment, the corresponding contribution to the field can be calculated analytically (Fig. 2 (d)).

Applying the fundamental theorem of the integral calculus to the integral appearing in (4), it is apparent that the magnetic field is a linear combination of functions, each of which being the channel base current waveform time-shifted by a quantity equal to the time necessary for the current to reach the beginning of the corresponding interval (details will be provided in the following section). The proposed approach makes it therefore possible to obtain an analytical expression for the lightning magnetic field over a perfectly conducting ground assuming the TL model (attenuation function equal to unity). On the other hand, no assumptions are made either on the channel-base current, which can be of any arbitrary waveshape, or on the value of the return stroke speed.

Once the magnetic field has been evaluated, the vertical and radial components of the lightning electric field can be evaluated analytically using Maxwell's equations, starting from the magnetic field formula without any further assumptions.

Finally, to account for the finite ground conductivity, the horizontal component of the electric field needs to be corrected using the equation proposed by Cooray and Rubinstein [26]. In [30], it is shown that, in the time domain, the Cooray-Rubinstein equation is the sum of two terms: the first one is proportional to the magnetic field over a perfectly-conducting ground, and the second is the solution of an Ordinary Differential Equation (ODE) with constant coefficients and a free term equal again to the magnetic field. If one develops the channel base current as a sum of exponential functions (see e.g., [40]), also the magnetic field can be written as a linear combination of exponential functions thanks to the proposed approach. This means that the ODE can be solved analytically, also resulting in an analytical expression for the horizontal electric field over a lossy ground with no other assumptions.

Note that, even if the method is characterized by the division of the channel into segments (which is a typical feature of a numerical method), we claim that the formula can be considered as analytical for the following reasons:

1) As will be detailed later on in the paper, a suitable change of variable allows one to perform the division of the channel into segments only once. As a consequence, the extremes of the segments and the parameters of the piecewise linear function approximating $\xi(z',t)$ are always the same (they are reported in the companion paper);
2) the coefficients of the linear combination are analytical functions of the return stroke speed;
3) the coordinates of the observation point appear in the above-mentioned time sequence in an analytical way;
4) the number of segments into which the channel has to be divided in order to obtain the desired accuracy is quite limited (see the analysis appearing in the companion paper).

## III. LIGHTNING ELECTROMAGNETIC FIELDS OVER AN IDEAL GROUND

The present section derives analytical expressions for the lightning electromagnetic fields over an ideal ground, starting with the magnetic field (subsection A) necessary to evaluate the electric field components (subsection B).

### A. *Magnetic field*

Let us reconsider (4). Introducing

$$\eta = \frac{z'-z}{r} \quad \text{and} \quad f(\eta) = \sqrt{1+\eta^2} \tag{5}$$

Equation (4) becomes:

$$\begin{aligned} H_\phi^{id}(r,z,t) &= \frac{c_0}{2\pi r v} i_0\left(t - \frac{R(0)}{c_0}\right) + \\ &+ \frac{c_0}{4\pi r}\left(\frac{1}{c_0^2} - \frac{1}{v^2}\right)\left[I_{image} + I_{source}\right] \end{aligned} \tag{6}$$

where

$$I_{image} = r\int_{-\frac{H+z}{r}}^{-\frac{z}{r}} i_0'\left(\xi_{r,z,t}^{i,0}(\eta)\right)\cdot 1\left(\xi_{r,z,t}^{i,0}(\eta)\right)d\eta \tag{7}$$

Having posed

$$\xi_{r,z,t}^{i,0}(\eta) = t - \frac{r}{c_0}f(\eta) + \frac{z+r\eta}{v} \tag{8}$$

And

$$\begin{aligned} I_{source} &= r\int_{-\frac{z}{r}}^{0} i_0'\left(\xi_{r,z,t}^{s,0}(\eta)\right)\cdot 1\left(\xi_{r,z,t}^{s,0}(\eta)\right)d\eta + \\ &+ r\int_{0}^{\frac{H-z}{r}} i_0'\left(\xi_{r,z,t}^{s,0}(\eta)\right)\cdot 1\left(\xi_{r,z,t}^{s,0}(\eta)\right)d\eta \end{aligned} \tag{9}$$

where

$$\xi_{r,z,t}^{s,0}(\eta) = t - \frac{r}{c_0}f(\eta) - \frac{z+r\eta}{v} \tag{10}$$

Please note that the superscripts *s* and *i* stands, respectively, for “source” and “image”.

Relation (9) is split into two parts, corresponding to positive and negative values of the integration variable. This will be useful in the following description of the procedure.

The main problem in solving analytically the integrals in (7) and (9) is that f is a nonlinear function of η. If this was not the case, one could i) solve analytically the inequalities expressing the condition for which the Heaviside function is equal to 1 and ii) perform a linear change of variable in the argument of $i_0$ for the calculation of the integral. The main idea of the method is to divide the channel and its image into *N*+1 intervals. Along each interval, f can be approximated by the secant passing through its extremes (see the Appendix for the details on the definition of a piecewise linear function g that approximates f). From here on, we will use the approximate formulas $\xi_{r,z,t}^{i}$ for $\xi_{r,z,t}^{i,0}$ and $\xi_{r,z,t}^{s}$ for $\xi_{r,z,t}^{s,0}$ obtained inserting the piecewise linear function g instead of *f* into (8). Please note that (5) has the advantage that the piecewise linear approximation g of the function f has to be made just once regardless of the position of the observation point. A graphical representation of the procedure related to the subdivision of the whole return stroke channel is shown in

Fig. 3, and an illustration of the relation between the function f and its piecewise linear approximation *g* is given in Fig. 7 in the Appendix.

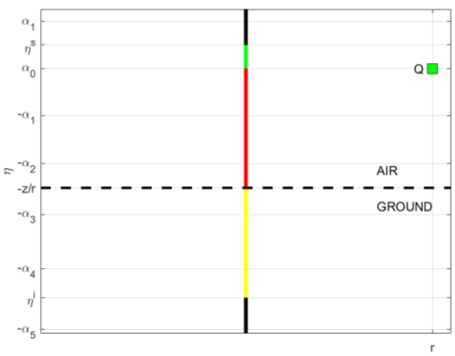


Fig. 3 Definition of the intervals into which the channel is divided. Yellow, red and green segments represent the portions of the image and of the source channel that contribute to the field at point *Q*(*r*,*z*) at time t. The red (green) part corresponds to the domain of the first (second) integral of (9), while the yellow part corresponds to the domain of the integral in (7) .The black segments are the portions of the channel whose fields have not yet reached the observation point *Q*.

Let us start with the calculation of the image channel contribution to the magnetic field (Eq. (7)). Due to the Heaviside function, (7) is not trivial if $\xi_{r,z,t}^{i}(\eta) > 0$ , which, at fixed *r* and *z*, defines the region depicted in Fig. 4.

First of all, one has to find the interval in which $x = -z/r$ lies, i.e. find $j^* \in \{1,\ldots,N\}$ such that $-\alpha_{j^*} \le -z/r \le -\alpha_{j^*-1}$ (in Fig. 3, $j^* = 3$ ). Please note that *x* should not be confused with the x-coordinate. Then the time instant $t_*$ before which the fields in *Q* are zero can be found as follows:

$$\xi_{r,z,t_*}^{i}\left(-\frac{z}{r}\right) = 0 \Rightarrow t_* = \frac{b_{j^*}}{c_0}r + \frac{a_{j^*}}{c_0}z \tag{11}$$

Moreover, the time instants $t_j^i$ for which the dipole at point $-\alpha_j$ contributes to the fields in *Q* are given by:

$$\xi_{r,z,t_j^i}^{i}\left(-\alpha_j\right) = 0 \Rightarrow t_j^i = \left(\frac{a_i\alpha_j + b_j}{c_0} + \frac{\alpha_j}{v}\right)r - \frac{1}{v}z \tag{12}$$

This defines the following sequence of time instants $t_N^i > \ldots > t_{j^*}^i \ge t_*$. Finally, for a fixed time $t \ge t_*$ the portion of the image that contributes to the fields in *Q* is given by solving the following inequality:

$$\xi_{r,z,t_j^i}^{i}(\eta) \ge 0 \Rightarrow -\frac{A_j^i}{r}\left(t - \frac{rb_j}{c_0} + \frac{z}{v}\right) = \eta^i \le \eta \le -\frac{z}{r} \tag{13}$$

where $A_j^i = \left(\frac{a_j}{c_0} + \frac{1}{v}\right)^{-1}$. At this point, (7) can be split into all the intervals that provide a contribution, partly or totally, to the fields.

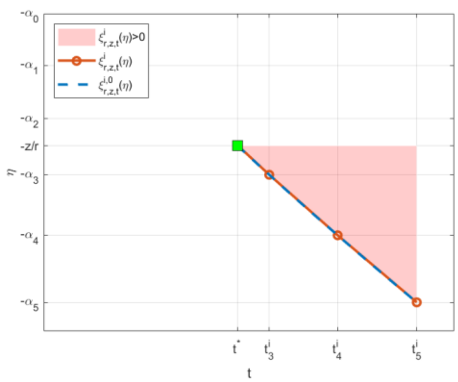

Fig. 4 View of the region $\xi^i_{r,z,t}(\eta) > 0$ for fixed values of r and z when j*=3.

The linearity of $\xi^i_{r,z,t}$ permits to solve the integral in (7) analytically as

$$I_{image} = A^i_{j^*} i_0\left(t - t_*\right) + \sum_{h=j^*}^{N-1}\left(A^i_{h+1} - A^i_h\right) i_0\left(t - t^i_h\right) \quad (14)$$

where the property that $i_0(t) = 0$ for $t < 0$, has been exploited. The physical interpretation of (14) is the following: If $t < t_*$, the propagation of the current along the channel and of the fields along the air is not sufficient to produce a nonzero magnetic field at point *Q*. If $t_* \le t < t^i_{j^*}$, only part of the first interval $\left[-\alpha_{j^*}, -\alpha_{j^*-1}\right]$ belonging to the image channel contributes to it. If $t^i_{j-1} \le t \le t^i_j$, all the intervals $\left[-\alpha_h, -\alpha_{h-1}\right]$ with $h \in \left\{j^*, \ldots, j-1\right\}$ and part of the interval $\left[-\alpha_j, -\alpha_{j-1}\right]$ produce a field at point *Q*.

Now, let us move to the source contribution in (9), which is not trivial if $\xi^s_{r,z,t}(\eta) > 0$. At fixed *r* and *z*, this condition defines the region depicted in Fig. 5 (in this case, $j^* = 3$).
As before, the following time sequence can be defined $t^{s,2}_N > \ldots > t^{s,2}_0 = t^{s,1}_0 > \ldots > t^{s,1}_{j^*-1} \ge t_*$, where $t^{s,2}_0 = t^{s,1}_0$ is the time at which the dipole placed at $\eta = 0$ starts contributing to the field and can be obtained by solving:

$$\xi^s_{r,z,t^{s,1}_0}(0) = 0 \Rightarrow t^{s,1}_0 = \frac{1}{c_0} r + \frac{1}{v} z \quad (15)$$

Similarly, the time instants $t^{s,1}_j$ (for $j = 1, \ldots, j^* - 1$ and $j^* > 1$) and $t^{s,2}_j$ (for $j = 1, \ldots, N$), necessary for the fields generated at points $-\alpha_j$ and $\alpha_j$ to reach point *Q*, are obtained, respectively, by solving:

$$\xi^s_{r,z,t^{s,1}_j}(-\alpha_j) = 0 \Rightarrow t^{s,1}_j = \left(\frac{a_j \alpha_j + b_j}{c_0} - \frac{\alpha_j}{v}\right) r + \frac{1}{v} z \quad (16)$$

and

$$\xi^s_{r,z,t^{s,2}_j}(\alpha_j) = 0 \Rightarrow t^{s,2}_j = \left(\frac{a_j \alpha_j + b_j}{c_0} + \frac{\alpha_j}{v}\right) r + \frac{1}{v} z \quad (17)$$

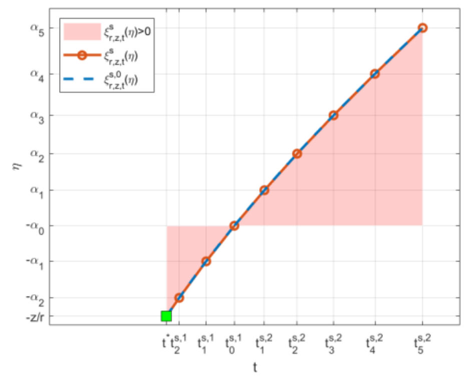

Fig. 5 View of the region $\xi^s_{r,z,t}(\eta) > 0$ for fixed values of *r* and *z* when $j^*$=3

Hence, for $t_* \le t \le t^{s,1}_0$, only part of the source channel with $\eta < 0$ contributes to the field at point *Q*. More precisely, it is given by solving the following inequality:

$$\xi^s_{r,z,t}(\eta) \ge 0 \Rightarrow -z/r \le \eta \le \eta^s = -\frac{A^{s,1}_j}{r}\left(t - \frac{rb_j}{c_0} - \frac{z}{v}\right) \quad (18)$$

where $j = \max\{h \in \left\{0, \ldots, j^* - 1\right\} : t \le t^{s,1}_k \ \forall k = 0, \ldots, h\}$ and $A^{s,1}_j = \left(\frac{a_j}{c_0} - \frac{1}{v}\right)^{-1}$.

Otherwise, if $t > t^{s,1}_0 = t^{s,2}_0$, all of the source channel with $\eta < 0$ and part of the source channel with $\eta > 0$ contribute to the field at point *Q*. Therefore, there exists $j \in \{1, \ldots, N\}$ such that $t^{s,2}_{j-1} \le t \le t^{s,2}_j$ and a portion of the source channel with $\eta > 0$ that produces a non-zero field at *Q* is given by solving the following inequality:

$$\xi^s_{r,z,t}(\eta) \ge 0 \Rightarrow 0 \le \eta \le \eta^s = \frac{A^{s,2}_j}{r}\left(t - \frac{rb_j}{c_0} - \frac{z}{v}\right) \quad (19)$$

where $A^{s,2}_j = \left(\frac{a_j}{c_0} + \frac{1}{v}\right)^{-1}$.

As in the case of the image contribution, the first integral in (9), (from now on $I_{source,1}$), can be analytically solved as follows (recall $i_0(t) = 0$ for $t < 0$)

$$I_{source,1} = A^{s,1}_{j^*} i_0\left(t - t_*\right) + \sum_{h=1}^{j^*-1}\left(A^{s,1}_{h+1} - A^{s,1}_h\right) i_0\left(t - t^{s,1}_h\right) + A^{s,1}_1 i_0\left(t - t^{s,1}_0\right) \quad (20)$$

The physical interpretation of formula (20) is the following:

If $t < t_*$, the propagation of the current along the channel and of the fields along the air is not sufficient to produce a nonzero magnetic field at point *Q*. If $t_* \le t \le t^{s,1}_{j^*-1}$, only part of the interval between the ground and the first point at which *g* is sampled contributes to the field at point *Q*. If $t^{s,1}_{j} < t \le t^{s,1}_{j-1}$, with $j = 1,..., j^* - 1$, all the intervals $[-\alpha_h, -\alpha_{h-1}]$ with $h \in \{2,..., j^* - 1\}$ and part of the interval $[-\alpha_1, 0]$ produce fields at point *Q*. Finally, if $t > t^{s,1}_0$, all of the source channel $[-z/r, 0]$ contributes to the magnetic field.

The second term in (9) (from now on $I_{source,2}$), is much easier to evaluate because this part of the channel is all in the air and all characterized by a positive value of the integral variable $\eta$. As a consequence, as before, one has:

$$I_{source,2} = A^{s,2}_1 i_0\left(t - t^{s,2}_0\right) + \sum_{h=1}^{N}\left(A^{s,2}_{h+1} - A^{s,2}_h\right) i_0\left(t - t^{s,2}_h\right) \quad (21)$$

and its physical interpretation is the same as before. Here too, the property $i_0(t) = 0$ for $t < 0$ has been used.

### B. *Electric field*

Maxwell's equations applied to the geometry of Fig. 1 allow to state that the vertical and radial components of the electric field can be calculated as [41]

$$\begin{aligned} E^{id}_r(t) &= -\mu_0 c_0^2 \int_0^t \frac{\partial H^{id}_\phi}{\partial z} d\tau \\ E^{id}_z(t) &= \mu_0 c_0^2 \int_0^t \frac{1}{r}\frac{\partial (r H^{id}_\phi)}{\partial r} d\tau \end{aligned} \quad (22)$$

It is sufficient to derive the magnetic field with respect to either *r* and *z* and to integrate along the time. The dependence on the spatial variables *r* and *z*, contained in the definition of the time instants (11)-(12), (15)-(16) and (17), is linear. This allows us to apply the derivation and integration rules for constant functions and to solve analytically the integrals in (22) as follows:

$$\begin{aligned} \int_0^t \frac{\partial H^{id}_\phi}{\partial z} d\tau &= \frac{-z}{2\pi r v R(0)} i_0\left(t - \frac{R(0)}{c_0}\right) + \\ &+ \frac{c_0}{4\pi r}\left(\frac{1}{c_0^2} - \frac{1}{v^2}\right)\left[\int_0^t \frac{\partial I_{image}}{\partial z} d\tau + \int_0^t \frac{\partial I_{source}}{\partial z} d\tau\right] \end{aligned} \quad (23)$$

and

$$\begin{aligned} \int_0^t \frac{1}{r}\frac{\partial (r H^{id}_\phi)}{\partial r} d\tau &= \frac{-1}{2\pi v R(0)} i_0\left(t - \frac{R(0)}{c_0}\right) + \\ &+ \frac{c_0}{4\pi r}\left(\frac{1}{c_0^2} - \frac{1}{v^2}\right)\left[\int_0^t \frac{\partial I_{image}}{\partial r} d\tau + \int_0^t \frac{\partial I_{source}}{\partial r} d\tau\right] \end{aligned} \quad (24)$$

Indicating with $\zeta$ the spatial variable ($\zeta \equiv r$ or $\zeta \equiv z$)

$$\int_0^t \frac{\partial I_{image}}{\partial \zeta} d\tau = -A^i_{j^*} \frac{\partial t_*}{\partial \zeta} i_0(t - t_*) - \sum_{h=j^*}^{N-1} (A^i_{h+1} - A^i_h) \frac{\partial t^i_h}{\partial \zeta} i_0(t - t^i_h) \quad (25)$$

$$\begin{aligned} \int_0^t \frac{\partial I_{source,1}(r,z,\tau)}{\partial \zeta} d\tau &= -A^{s,1}_{j^*} \frac{\partial t_*}{\partial \zeta} i_0(t - t_*) + \\ &+ \sum_{h=1}^{j^*-1} (A^{s,1}_{h+1} - A^{s,1}_h) \frac{\partial t^{s,1}_h}{\partial \zeta} i_0(t - t^{s,1}_h) + A^{s,1}_1 \frac{\partial t^{s,1}_0}{\partial \zeta} i_0(t - t^{s,1}_0) \end{aligned} \quad (26)$$

$$\begin{aligned} \int_0^t \frac{\partial I_{source,2}(r,z,\tau)}{\partial \zeta} d\tau &= -A^{s,2}_1 \frac{\partial t^{s,2}_0}{\partial \zeta} i_0(t - t^{s,2}_0) + \\ &- \sum_{h=1}^{N-1} (A^{s,2}_{h+1} - A^{s,2}_h) \frac{\partial t^{s,2}_h}{\partial \zeta} i_0(t - t^{s,2}_h) \end{aligned} \quad (27)$$

The derivatives contained in the above formulas are all constant and they can be easily calculated by way of (11)-(12), (15)-(16), and (17).

## IV. Cooray-Rubinstein Formula

In [30], it was shown that the finite conductivity of the ground can be accounted for if the ideal horizontal electric field is added to the term:

$$E_{CR}(r,z,t) = -\eta H^{id}_\phi(r,0,t) - \eta \sum_{k=1}^{N_{RA}} r_k\, \mathcal{X}_k(t) \quad (28)$$

where $\chi_k$ is the solution of the linear differential equation:

$$\begin{cases} \dfrac{d\mathcal{X}_k}{dt} = \dfrac{c_k}{\tau_G}\, \mathcal{X}_k + \dfrac{1}{\tau_G} H^{id}_\phi(r,0,t) \\ \mathcal{X}_k(0) = 0 \end{cases} \quad (29)$$

The coefficients $c_k$ and $r_k$ appear in Tab. 1 of [30], $N_{RA}$=12 and $\tau_G = \varepsilon / \sigma$. The time domain analytical expression for the Cooray-Rubinstein formula can be obtained if one solves (29). Adopting the previously developed formulas to calculate the magnetic field at ground level, it readily follows that $I_{image}(r,0,t) = I_{source,2}(r,0,t)$ and $I_{source,1}(r,0,t) = 0$. Thus:

$$H^{id}_\phi(r,0,t) = \begin{cases} 0 & t \le \dfrac{r}{c_0} \\ \displaystyle\sum_{j=0}^{N-1} B_j i_0\left(t - t^{s,2}_{j|z=0}\right) & t > \dfrac{r}{c_0} \end{cases} \quad (30)$$

Where, for the sake of simplicity, we define

$$\begin{aligned} B_0 &= \frac{c_0}{2\pi r v} + \frac{c_0}{2\pi r}\left(\frac{1}{c_0^2} - \frac{1}{v^2}\right) A^{s,2}_1 \\ B_j &= \frac{c_0}{2\pi r}\left(\frac{1}{c_0^2} - \frac{1}{v^2}\right)(A^{s,2}_{j+1} - A^{s,2}_j) \qquad j = 1, \ldots, N-1 \end{aligned} \quad (31)$$

and $t^{s,2}_{j|z=0}$ is obtained by (15) and (17) evaluated for $z = 0$, i.e.,

$$t^{s,2}_{j|z=0} = \left(\frac{a_j \alpha_j + b_j}{c_0} + \frac{\alpha_j}{v}\right) r \quad \text{and} \quad t^{s,2}_{0|z=0} = \frac{r}{c_0} \quad (32)$$

The analytical solution of (29) reads:

$$\mathcal{X}_k(t) = \frac{1}{\tau_G} e^{\frac{c_k}{\tau_G}t} \int_{\tau=0}^{t} e^{-\frac{c_k}{\tau_G}\tau} H_\phi^{id}(r,0,\tau)\,d\tau \tag{33}$$

Assuming now that the channel-base current can be expressed as the sum of $N_G$ exponential terms of the kind:

$$i_0(t) = \sum_{h=1}^{N_G} q_h\, e^{s_h t}\, 1(t) \tag{34}$$

and inserting (34) into (30), the integral in (33) can be solved analytically and one gets:

$$\mathcal{X}_k(t) = \frac{1}{\tau_G} e^{\frac{c_k}{\tau_G}t} \sum_{j=0}^{N-1} \sum_{h=1}^{N_G} D_{hj} G_{jhk}(t) \tag{35}$$

where

$$D_{hj} = q_h B_j\, e^{-s_h t_{j|z=0}^{s,2}} \tag{36}$$

and

$$G_{jhk}(t) = \begin{cases} 0 & t \le t_{j|z=0}^{s,2} \\ \dfrac{1}{F_{hk}}\left[ e^{F_{hk}t} - e^{F_{hk}t_{j|z=0}^{s,2}} \right] & t > t_{j|z=0}^{s,2} \end{cases} \tag{37}$$

with

$$F_{kh} = -\frac{c_k}{\tau_G} + s_h \tag{38}$$

Please note that (34) is reasonable for the most widely adopted expressions for the channel base current. It obviously holds true if one uses the Double EXPonential (DEXP [42]) and only 9 terms are required to properly represent the Heidler's current [43], by using the Prony's expansion (see [40] for details).

## V. Summary

This section aims at providing the reader with a summary of the method and of the steps that one has to follow in order to correctly apply it.

The step-by-step procedure is presented here. A graphical representation can be found in Fig. 6.

1. Divide the channel according to the method proposed in the Appendix and evaluate the points $\alpha_j$ and the coefficients $a_j, b_j$.
2. Compute the coefficients $A_j^i$ , $A_j^{s,1}$ and $A_j^{s,2}$ according to (13), (18) and (19).
3. Calculate $t_*$ using (11), $t_j^i$ using (12), $t_0^{s,1} = t_0^{s,2}$. using (15), $t_j^{s,1}$ using (16), and $t_j^{s,2}$ using (17).
4. To calculate the magnetic field, apply (14) to evaluate the integral in (7) and (20)-(21) to evaluate the two integrals in (9).
5. Derive the magnetic field with respect to *r* and *z* and integrate along the time (equations (22)-(27)) to obtain the radial electric field and the vertical electric field in case of a PEC ground.
6. Apply the Prony's expansions in order to express the channel-base current as a sum of exponentials with (34).
7. Insert the result of the previous step into (30) and apply (35)-(38) and (28) in order to obtain the radial electric field in the case of a finitely conducting ground.

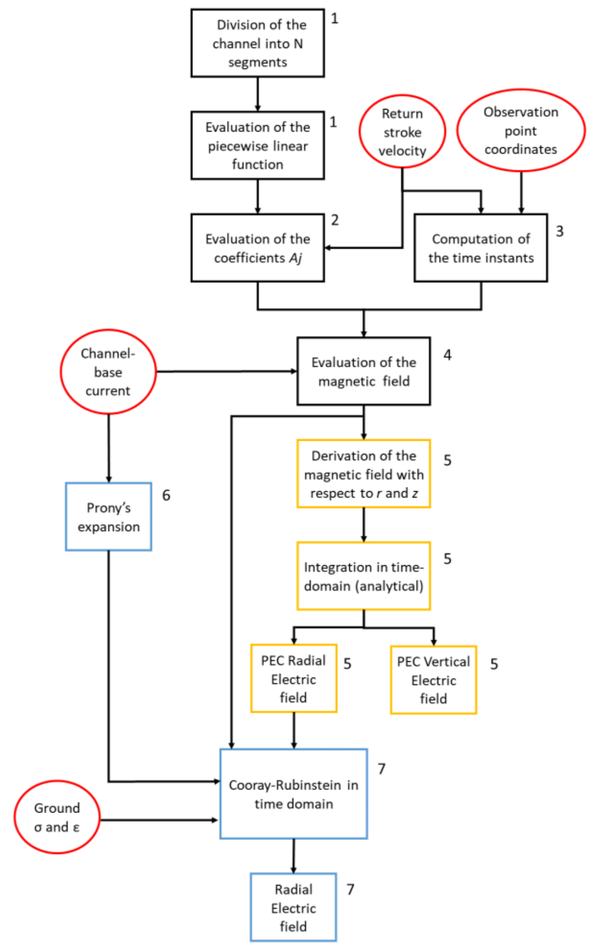


Fig. 6 Graphical representation of the step-by-step procedure

## VI. Conclusions

This paper presented analytical formulas to evaluate the lightning electromagnetic fields generated by an arbitrary channel base current under the hypothesis that the attenuation function of the return stroke current model is P(z')=1 (TL model). The vertical electric field and the azimuthal magnetic field components are derived under the assumption of a perfectly conducting ground. The finite ground conductivity is accounted for in the derivation of the horizontal electric field using the time-domain Cooray-Rubinstein equation. The main idea is based on the division of the channel into segments. For each one of them, the distance between the observation point and the source point can be approximated with a linear function of their coordinates. This allows firstly to obtain an analytical expression for the magnetic field and then for the vertical and radial components of the electric field over an ideal ground. Finally, an analytical expression of the Cooray-Rubinstein formula in the time domain is proposed to account for the effects of the finite conductivity of the ground.

Compared to existing analytical formulations, the derived formulas have the advantage that no assumptions are made either on the channel-base current which can be of any arbitrary waveshape, or on the value of the return stroke speed which can take any value. Furthermore, in the derived equations, the finite ground conductivity is accounted for in the computation of the radial electric field.

In the companion paper, simulation results will be presented that validate the proposed approach against classical numerical integration, highlighting very good accuracy and a meaningful reduction in the computational effort.

## VII. Appendix: Piecewise Linear Approximation of the Distance Between an Elementary Channel Segment and the Observation Point

If $A>0$ and $f:[0,A]\quad \eta\mapsto\sqrt{1+\eta^2}\in[0,+\infty)$, the approximation of *f* by the secant passing through *N*+1 points can be achieved introducing a piecewise linear function *g*,

$$g(\eta)=\begin{cases} a_1|\eta|+b_1 & \alpha_0\le\eta\le\alpha_1 \\ \vdots & \\ a_N|\eta|+b_N & \alpha_{N-1}\le|\eta|\le\alpha_N \\ a_{N+1}|\eta|+b_{N+1} & \alpha_N\le|\eta|\le\alpha_{N+1} \end{cases} \tag{A1}$$

where $\alpha_0=0$ and $\alpha_{N+1}=A$. Imposing $f(\alpha_j)=g(\alpha_j)$ for any $j=1,\ldots,N+1$, one has

$$a_j=\frac{f(\alpha_j)-f(\alpha_{j-1})}{\alpha_j-\alpha_{j-1}} \quad \text{and} \quad b_j=f(\alpha_j)-a_j\alpha_j \tag{A2}$$

Let us observe that from a practical viewpoint, the parameter *A* should be chosen so that *g* approximates *f* along all the portion of the channel that contributes to the field in the desired time window. Details on the choice of *A* will be provided in the companion paper. A schematic representation of *g* with respect to *f* is proposed in Fig.2.

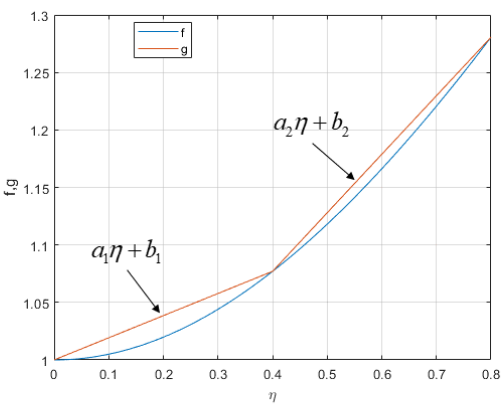


Fig. 7 Schematic plot of the piecewise linear function *g* and of the function *f*. Please not that *f* is even, consequently only the positive part is represented.

Since $f$ is a convex function, for any $j=1,\ldots,N+1$, it follows that $f(\eta)\le g(\eta)$ for $\eta\in[\alpha_{j-1},\alpha_j]$. Then one can prove the following two properties:

Theorem 1: For any $j=1,\ldots,N+1$

$$0<a_j<1 \tag{A3}$$

and

$$\frac{a_j}{c_0}-\frac{1}{v}<0 \quad \text{and} \quad \frac{a_j}{c_0}+\frac{1}{v}>0 \tag{A4}$$

*Proof.* Since $f$ is increasing, the first inequality is obvious. As far as the second is concerned, using the definition in (A2) one has $\sqrt{1+\alpha_j^2}-\sqrt{1+\alpha_{j-1}^2}<\alpha_j-\alpha_{j-1}$. Taking the square of both (positive) members, one gets $1+\alpha_{j-1}\alpha_j<\sqrt{1+\alpha_j^2}\sqrt{1+\alpha_{j-1}^2}$ and taking again the square, one obtains $(\alpha_{j-1}-\alpha_j)^2>0$, which holds true as $\alpha_{j-1}\ne\alpha_j$ Moreover, recalling that $v\le c_0$, inequalities (A4) immediately follow

Theorem 2: For any $j=1,\ldots,N+1$, the function $H_j(\eta)=(a_j\eta+b_j)-f(\eta)$ reaches its maximum value in $[\alpha_{j-1},\alpha_j]$ at

$$\eta_j=\frac{a_j}{\sqrt{1-a_j^2}} \tag{A5}$$

*Proof.* Note that $H_j(\alpha_j)=H_j(\alpha_{j-1})=0$ and $H_j\in C^1([\alpha_{j-1},\alpha_j])$. The Weierstrass and Rolle theorems ensure that the maximum exists and it corresponds to the point $\eta_j$ for which the first derivative of $H_j$ is zero.

The problem one has to face at this point is the following: given a number of points *N*, define a criterion on how to choose such points to guarantee the best approximation of function *f*. For each interval $[\alpha_{j-1},\alpha_j]$, a way to obtain the best approximation is to minimize the maximum value of function $H_j$ (that is to say, to minimize the maximum difference between function $f$ and its linear approximation). So, the following constrained optimization problem can be set up. Find $(\alpha_1,\alpha_2,\ldots,\alpha_N)$ that minimize

$$\sum_{j=1}^{N+1}H_j\left(\frac{a_j}{\sqrt{1-a_j^2}}\right) \tag{A6}$$

s.t.

$$0<\alpha_1<\alpha_2<\ldots<\alpha_N<A \tag{A7}$$

and

$$a_j=\frac{\sqrt{1+\alpha_j{}^2}-\sqrt{1+\alpha_{j-1}{}^2}}{\alpha_j-\alpha_{j-1}} \tag{A8}$$

In the companion paper, a criterion to select $N$ as a compromise between numerical effort and accuracy is proposed. Moreover, the corresponding values of $\alpha_i$ will be provided.

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