Relativistic Doppler effect in an extending transmission line: Application to lightning

Abdolhamid Shoory,[1] Farhad Rachidi,[1] and Marcos Rubinstein[2]

Received 2 November 2010; revised 24 March 2011; accepted 7 April 2011; published 6 July 2011.

[1] We present in this paper a thorough analysis of current wave propagation with arbitrary speed along an extending transmission line. We derive rigorous analytical equations in the time and frequency domains expressing the reflections of the current wave occurring at the extending end of the line. The derived equations reveal that it is not possible to represent current reflections occurring at the extending end of a transmission line using a constant, frequency-independent reflection coefficient, as previously done in the literature. The reflected wave from the extending end of the line is shown to be affected by the Doppler frequency shift. In other words, the reflected wave from an extending transmission line suffers distortion, the amount of which depends on the incident wave form, its frequency content, and the speed of the extending end of the line. The derived expression is in agreement with the relativistic Doppler effect and is consistent with the Lorentz transformation. Finally, engineering models for return strokes are generalized and closed-form analytical expressions are derived for the spatial-temporal distribution of the current along the channel accounting for reflections at ground and at the return stroke wave front taking into account the Doppler effect.

Citation: Shoory, A., F. Rachidi, and M. Rubinstein (2011), Relativistic Doppler effect in an extending transmission line: Application to lightning, *J. Geophys. Res.*, *116*, D13205, doi:10.1029/2010JD015279.

1. Introduction and Background

[2] Lightning return strokes models can be classified, depending on the type of governing equation, into four classes of models [*Rakov and Uman*, 1998], namely, (1) gas dynamic models, (2) electromagnetic models, (3) distributed-circuit models, and (4) engineering models.

[3] Among these classes of models, the engineering models have been extensively used since the 1940s to study electromagnetic radiation from lightning return strokes (see *Nucci et al.* [1990] for a review). In these models the spatial and temporal distribution of the channel current (or the channel charge density) is specified as a function of the current at the channel base, the return stroke speed, and a number of adjustable parameters [*Thottappillil et al.*, 1997; *Rakov and Uman*, 1998]. These models include the Bruce-Golde model [*Bruce and Golde*, 1941], the traveling current source (TCS) model [*Heidler*, 1985], the transmission line (TL) model [*Uman and McLain*, 1969], the modified transmission line with exponential current decay with height (MTLE) model [*Nucci et al.*, 1988; *Rachidi and Nucci*, 1990], the modified transmission line with linear current decay with height (MTLL) model [*Rakov and Dulzon*, 1987], and the Diendorfer and Uman (DU) model [*Diendorfer and Uman*, 1990] and its modified version (MDU) [*Thottappilil et al.*, 1991]. Recent developments on lightning return stroke models can be found in the work of *Rakov and Rachidi* [2009].

[4] The engineering models can be grouped in two categories, the lumped-source (also referred to as transmission-line-type or current-propagation) models and the distributed-source (also referred to as traveling-current-source-type or current-generation) models [*Rakov and Rachidi*, 2009]. *Cooray* [2003] showed that any lumped-source (LS) model can be formulated in terms of sources distributed along the channel and progressively activated by the upward moving return stroke front, as previously demonstrated for the MTLE model by *Rachidi and Nucci* [1990]. None of the engineering models in their original formulations considered the possibility of current reflections from the extending return stroke wave front. (This assertion applies to all classes of return stroke models.)

[5] For lightning strikes to tall towers, possible current reflections from the return stroke wave front have been considered in a few studies [e.g., *Janischewskyj et al.*, 1998; *Shostak et al.*, 1999; *Shostak et al.*, 2000; *Napolitano and Nucci*, 2009; *Mosaddeghi et al.*, 2010]. Specifically, *Shostak et al.* [2000] extended the MTLE model for the modeling of the lightning attachment to the 553-tall CN tower assuming a constant current reflection coefficient of −0.9 at the moving front of the lightning return stroke. These studies have shown that taking into account possible reflections at the return stroke wave front results in a better reproduction of the fine structure of the lightning current and radiated electromagnetic fields

[1]Electromagnetic Compatibility Laboratory, Swiss Federal Institute of Technology, Lausanne, Switzerland.
[2]Department of Electrical Engineering and Computer Science, University of Applied Sciences of Western Switzerland, Yverdon-les-Bains, Switzerland.

0148-0227/11/2010JD015279

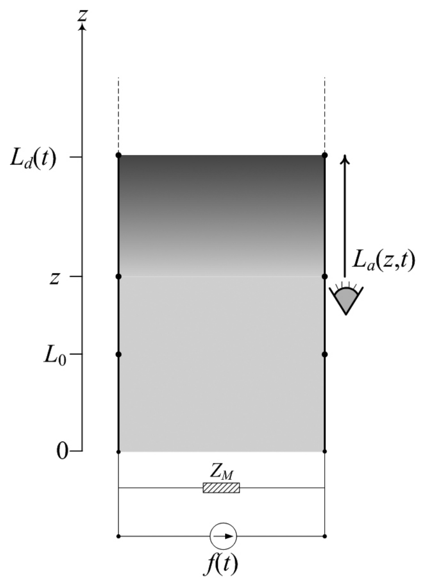

Figure 1. An extending transmission line along the z axis. (Z_M is the matching impedance). $L_a(z, t)$ is the apparent length of the line seen by an observer (shown with an eye symbol) at spatial position z and at time t and $L_d(t)$ is the dynamic spatial position of the upper end of the line at any time t.

associated with strikes to the CN Tower. Note that an incomplete reflection at the return stroke wave front indicates implicitly that part of the current is transmitted on to the leader region above the front.

[6] Further, the classical reflection mechanism employed in these studies is only valid when the return stroke wave front is stationary or at least moving with speeds significantly smaller than the speed of upward waves transmitted into the channel due to transient phenomena inside the tower. In an attempt to give a more realistic account of the boundary conditions at the moving return stroke wave front, *Heidler and Hopf* [1994a, 1994b] derived an expression for the current reflection coefficient using the TCS model and considering ground-initiated lightning return strokes. The derived expression for the current reflection coefficient is solely a function of the return stroke speed and the speed of light ($(v - c)/(v + c)$). The same expression for the current reflection coefficient was later used by *Schulz and Diendorfer* [1995] who proposed an extended version of the DU return stroke model [*Diendorfer and Uman*, 1990] to calculate radiated fields at different distances from the lightning channel. Interestingly, their simulation results show that the current was discontinuous at the return stroke wave front.

[7] More recently, *Mosaddeghi et al.* [2010] presented an extension of the engineering return stroke models for lightning strikes to tall structures that takes into account the presence of reflections at the return stroke wave front and the presence of an upward connecting leader. They used a similar approach as in the work of *Shostak et al.* [2000] but using the expression of *Heidler and Hopf* [1994a, 1994b] for the current reflection coefficient at the return stroke wave front. Simulation results for the magnetic fields were compared with experimental wave forms associated with lightning strikes to the CN Tower (553 m) and the predictions taking into account reflections at the wave front and the presence of upward connecting leaders were found to be in better agreement with experimental observations. *Raysaha et al.* [2010] presented an analysis taking into account nonlinear channel dynamics and corona effects along the channel. Their results suggest that the transmitted waves from the tall tower to the channel undergo significant attenuation in the region near the return stroke wave front resulting in negligible reflection.

[8] In this paper, we will present a rigorous analysis of wave propagation along an extending transmission line with an arbitrary speed. The analysis will then be applied to lightning return stroke modeling. The presence of elevated strike objects and/or upward connecting leaders is disregarded in the present analysis.

[9] The paper is organized as follows. In section 2 we will present a theoretical analysis of the current wave reflection from an extending transmission line both in the time domain (section 2.1) and in the frequency domain (section 2.2). The proposed formulation will be examined from the point of view of the relativistic Doppler effect in section 2.3. Section 3 will present the extension of the engineering return stroke models taking into account reflections at the extending return stroke wave front. The extension is based on the distributed source representation of the engineering models which allows a straightforward inclusion of reflections at both ends of the return stroke channel (ground level and the return stroke wave front). A discussion will be provided in section 4 and, finally, a summary and conclusions will be given in section 5.

2. Pulse Propagation in an Extending Transmission Line

[10] Consider a lossless transmission line along the z axis as shown in Figure 1. The lower termination of the line is fixed at $z = 0$. It has an initial length of L_0 at time $t = 0$ and lengthens upwards along the positive z axis with a constant speed v. The basic problem of the radiation from such a line, but with a static upper end and a square pulse waveform was studied by *Rubinstein and Uman* [1991]. In this paper, an arbitrary current source with a waveform $f(t)$ excites the line at its bottom end. The wave form propagates up along the line with a speed c (note that although we have used c to represent the speed of the upward moving wave front in the presented derivation, it does not have to be the speed of light; it only has to be greater than v; however, in this context, it is usually assumed to be the speed of light) greater than v and will eventually catch up with the moving upper end of the line (assumed to be an open-circuit). The wave will then be reflected and it will begin to propagate back down the line with the same speed c. For the sake of simplicity we will ignore in this section any reflections from the lower end of the line. In other words, we assume a perfectly matched termination at that end obtained by using the line's characteristic impedance. Such reflections will be considered in section 3 where the proposed formulation will be applied to lightning return stroke modeling. Further, we assume that the wave form suffers no distortion as it propagates up and down along the line. Two different deriva-

tions will be given in the following subsections, the first in the time domain and the second in the frequency domain.

2.1. Time Domain Derivation

[11] We will follow an instantaneous value $f(t_0)$ in the wave form $f(t)$ as it travels up the line, passes a spatial point z, and comes back to this point after being reflected at the extending termination of the line. The spatial position of this instantaneous value $f(t_0)$ at any time t until it catches up with the upper end of the line is given by

$$L_p(t) = c(t - t_0)u(t - t_0) \tag{1}$$

where $u(t)$ is the unit Heaviside step function and the subscript p indicates the length along the line traversed by the spatial position of the instantaneous value $f(t_0)$. On the other hand, the dynamic spatial position of the upper end of the line at any time t can be written as

$$L_d(t) = L_0 + vtu(t) \tag{2}$$

The height at which the two spatial positions given by (1) and (2) are identical is in fact the encounter point of the upward moving wave form instant and the extending end of the line, which can be obtained by solving the following equation for the encounter time t

$$c(t - t_0)u(t - t_0) = L_0 + vtu(t) \tag{3}$$

Since the encounter will happen necessarily at a time $t > t_0$, we can rewrite (3) dropping out the step functions as follows:

$$c(t - t_0) = L_0 + vt \tag{4}$$

from which we can solve for the time t at which the considered instantaneous value $f(t_0)$ of the wave form $f(t)$ reaches the top of the line,

$$t = \frac{L_0 + ct_0}{c - v} \tag{5}$$

Although t represents the catch-up time and thus is a particular value for this variable, we purposely did not include a subscript to avoid the particularization, since such a time exists for any instant in the exciting wave form.

[12] On the other hand, solving (4) or (5) for t_0 gives

$$t_0 = \frac{t(c - v) - L_0}{c} \tag{6}$$

The instantaneous incident wave form seen at the upward extending end of the line, $f_i(L_d(t), t)$ is therefore given by

$$f_i(L_d(t), t) = f(t_0) = f\left(\frac{t(c - v) - L_0}{c}\right) \tag{7}$$

This function will produce the right value of the exciting wave form reaching the moving front of the line which is generally applicable to the whole wave form. It is this function that needs to be reflected back from the top of the extending line. Assuming a complete reflection (open-circuit condition), we get the reflected wave form at the line top $L_d(t)$ as follows

$$f_r(L_d(t), t) = -f\left(\frac{t(c - v) - L_0}{c}\right) \tag{8}$$

To find the reflected wave form at an arbitrary spatial point z along the line, we can proceed as we did to calculate the time dependence of the wave form at the catch-up point. In doing so, we take (5), which expresses the time at which an instantaneous value $f(t_0)$ of the wave form $f(t)$ launched at instant t_0 from the bottom meets with the moving front. We will now add to that time the interval needed for the reflected wave to reach the observation point z. To obtain this interval, we first use (2) and (5) to calculate the height of the encounter

$$H_e = L_d(t) = L_0 + v\frac{L_0 + ct_0}{c - v} \tag{9}$$

The time interval required for the reflected wave to travel from this height to point z at the speed c is then given by

$$\Delta t = \frac{H_e - z}{c} = \frac{L_0 + v\frac{L_0 + ct_0}{c - v} - z}{c} \tag{10}$$

The total time that the instantaneous value at $t = t_o$ takes to travel from the base of the line up to the upward moving end and back down to position z (using as a reference the time $t = 0$ at which the excitation at the bottom of the line is initiated) is given by the sum of (5), which is the elapsed time until the upward excitation reaches the rising end of the line and (10) which is the time from that encounter point back down to position z:

$$t = \frac{L_0 + ct_0}{c - v} + \frac{L_0 + v\frac{L_0 + ct_0}{c - v} - z}{c} \tag{11}$$

Solving (11) for t_0, we get

$$t_0 = \frac{c - v}{c + v}\left(t + \frac{z}{c}\right) - \frac{2L_0}{c + v} \tag{12}$$

which represents the instant of the exciting wave form $f(t)$ seen at position z at any time t due to the first reflection off the moving top end of the line. Note that by using "instant" we mean the time corresponding to a given point in the exciting wave form $f(t)$.

[13] The reflected wave form at position z is therefore given by the exciting wave form $f(t)$ evaluated at the time t_0 given by (12),

$$f_r(z, t) = -f(t_0) = -f\left(\frac{c - v}{c + v}\left(t + \frac{z}{c}\right) - \frac{2L_0}{c + v}\right) \tag{13}$$

On the other hand, the incident wave form at point z and time t is simply the retarded value of the exciting wave form given by

$$f_i(z, t) = f\left(t - \frac{z}{c}\right) \tag{14}$$

The total wave form at spatial point z can then be obtained by superposition, adding the incident and reflected wave forms given by (14) and (13), respectively, as follows:

$$f_t(z, t) = f\left(t - \frac{z}{c}\right) - f\left(\frac{c - v}{c + v}\left(t + \frac{z}{c}\right) - \frac{2L_0}{c + v}\right) \tag{15}$$

It can be seen from (15), however, that the reflected wave form does undergo distortion as a result of the reflection at the extending end of the transmission line. Equation (15) can be compared in fact with Bergeron equations [*Tesche et al.*, 1997], with the only difference that Bergeron equations apply to stationary lines. The presence of an extending end, however, results in the dispersive factor $(c - v)/(c + v)$.

[14] In section 2.2 we present an equivalent frequency domain analysis, and we show that this distortion is actually the Doppler effect. Note that in any stage of the development of the above formulation, the causality of the wave forms should be maintained. In other words, any resulting wave form cannot have nonzero values before the onset of the exciting wave form at time $t = 0$.

2.2. Frequency Domain Derivation

[15] Let us assume that the exciting wave form at the bottom end of the line is a sinusoid of constant amplitude A_i and frequency ω given by $A_i e^{j\omega t}$. Then, with reference to Figure 1, the incident wave form at spatial position z due to this excitation can be written as

$$F_i(z,t,\omega) = A_i e^{j\omega t} e^{-j\omega\frac{z}{c}} \tag{16}$$

Assuming that the apparent length of the line seen by an observer at spatial position z and at time t is $L_a(z, t)$, we can write

$$t = \frac{L_a(z,t) - L_0}{v} + \frac{L_a(z,t) - z}{c}, \qquad L > z \tag{17}$$

The first term in (17) is the time delay taken by the line to extend from its initial length L_0 to its apparent length $L_a(z, t)$ and the second term is the retardation time from the extending end of the line to the observation point at z. Solving (17) for $L_a(z, t)$ yields

$$L_a(z,t) = \frac{vc}{v+c}\left(t + \frac{L_0}{v} + \frac{z}{c}\right) \tag{18}$$

Note that the apparent length given by (18) is clearly different from the dynamic length given by (2) and also the spatial position of the instantaneous value $f(t_0)$ in the incident wave form given by (1). The incident wave given by (16) travels up along the line, passes point z, and reflects back from the extending end of the line to the position z. The reflected wave seen by the observer at such a position can then be written as

$$F_r(z,t,\omega) = A_r e^{j\omega t} e^{-j\omega\frac{La(z,t)}{c-v}} e^{-j\omega\frac{La(z,t)-z}{c}} \tag{19}$$

Assuming a complete reflection at the moving end of the line (open-circuit condition), the total wave, which is the sum of the incident and reflected waves, should vanish at this end, i.e.,

$$F_i(L_a(z,t),t,\omega) + F_r(L_a(z,t),t,\omega) = 0 \tag{20}$$

Inserting (16) and (19) into (20) and after straightforward mathematical manipulations, we obtain

$$A_r = -A_i e^{j\omega\frac{v}{c(c-v)}L_a(z,t)} \tag{21}$$

Replacing (21) into (19), replacing $L_a(z, t)$ from (18), and again after straightforward mathematical manipulations, we obtain

$$F_r(z,t,\omega) = -A_i e^{j\omega\frac{c-v}{c+v}\left(t+\frac{z}{c}\right)} e^{-j\omega\frac{2L_0}{c+v}} \tag{22}$$

The total wave at spatial position z can then be written as the sum of the incident (16) and reflected (22) waves as follows:

$$F_t(z,t,\omega) = A_i e^{j\omega\left(t-\frac{z}{c}\right)} - A_i e^{j\omega\left(\frac{c-v}{c+v}\left(t+\frac{z}{c}\right)-\frac{2L_0}{c+v}\right)} \tag{23}$$

Equation (23) was obtained assuming a single frequency harmonic excitation at the bottom end of the line. It is clear that since any given wave form in the time domain can be represented using its Fourier transform, equation (23) can be easily transformed into the time domain to give equation (15) for a general time domain excitation, namely $f(t)$.

2.3. Relation to Relativistic Doppler Effect

[16] It can be readily seen from (23) that the reflected wave from the extending end of the line has a frequency that is shifted in spectrum from the source frequency. This is the so-called Doppler effect usually understood in its classical form [*Cheng*, 1983]. To explain such a frequency shift from a relativistic Doppler effect point of view and to show that it is consistent with Lorentz transformation, we first fix the source emitting a signal with frequency ω at $z = 0$ and let the observer move with the extending end of the line at speed v. According to the relativistic Doppler effect (see, for instance, chapter 11 of *Jackson* [1999]), the observer receives the source signal at a different frequency given by

$$\omega_{o1} = \omega\sqrt{\frac{c-v}{c+v}} \tag{24}$$

Now let us assume another observer located at $z = 0$. The extending end of the line, after receiving the incident wave and transmitting it back through reflection acts as another source emitting a signal with frequency ω_{o1} toward this observer. Since this source is again moving away from the observer with speed v, the received frequency by the observer at $z = 0$ can then be written as

$$\omega_{o2} = \omega_{o1}\sqrt{\frac{c-v}{c+v}} = \omega\frac{c-v}{c+v} \tag{25}$$

which is the frequency derived in (23).

3. Revision of Return Stroke Models

[17] In this section, the formulation we developed for the Doppler effect in an extending transmission line will be used to revise engineering return stroke models taking into account the reflections from the return stroke wave front. We will first consider the MTLE model for ground initiated lightning return strokes and then we will generalize the formulation to other models.

[18] The spatial-temporal distribution of the return stroke current along a vertical channel (see Figure 2) according to the MTLE model [*Nucci et al.*, 1988; *Rachidi and Nucci*, 1990] is given by

$$i(z,t) = e^{-\frac{z}{\lambda}} i\left(0, t - \frac{z}{v}\right) u\left(t - \frac{z}{v}\right) \tag{26}$$

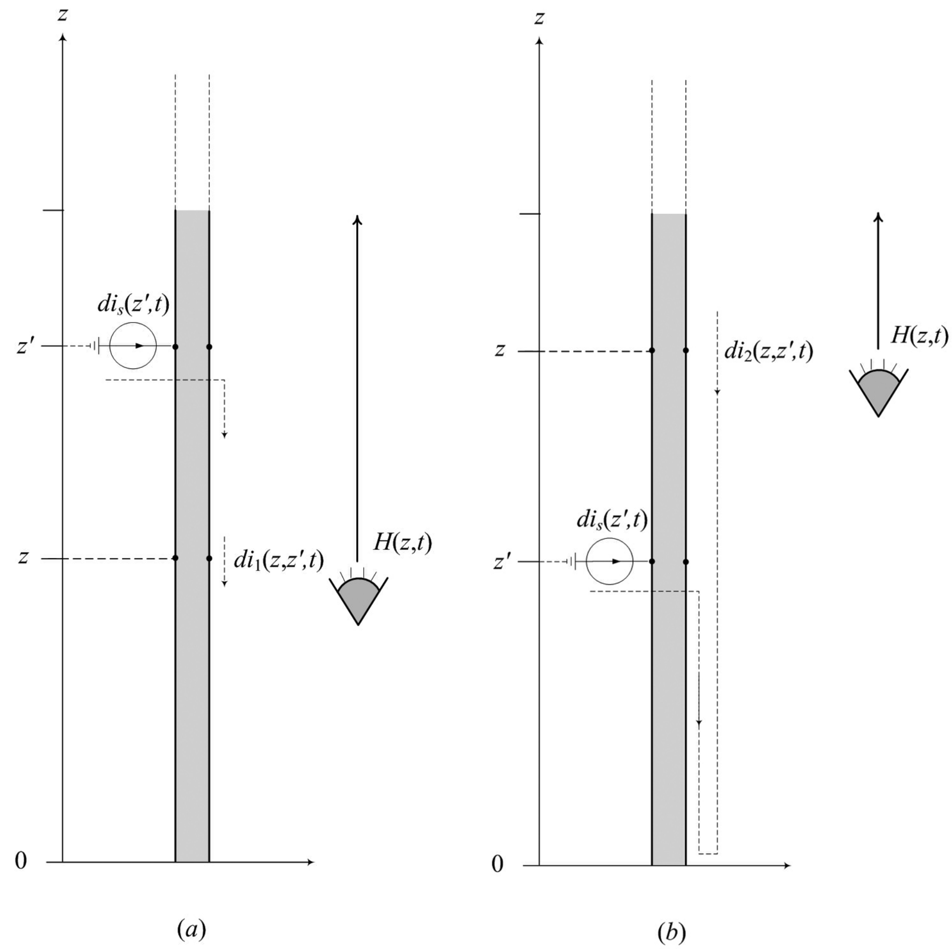

Figure 2. Distributed-source representation of the lightning channel in engineering return stroke models for the case of no strike object but considering reflections at ground (adopted from *Rachidi et al.* [2002]). (a) The current source located at z' is above the observation point at z and (b) the current source located at z' is below the observation point at z.

where z is the height above the ground, λ is the attenuation height constant, $i(0,t)$ is the current at the channel base, and v is the return stroke speed assumed to be constant. The spatial-temporal distribution of the current (26) can be viewed as being due to the contribution of distributed sources along the channel [*Rachidi and Nucci*, 1990]. Each source is switched on when the return stroke wave front reaches its altitude and delivers a current which flows down the channel at the speed of light c. The general expression for such current source located at height z' is given by *Rachidi and Nucci* [1990] as

$$di_s(z',t) = \begin{cases} 0 & t < \frac{z'}{v} \\ g\left(t - \frac{z'}{v}\right)e^{-\frac{z'}{\lambda}}dz' & t \geq \frac{z'}{v} \end{cases} \quad (27)$$

where $g(t)$ can be an arbitrary function. Assuming a complete match at the channel base similar to the situation shown in Figure 1 in which the line is connected to its characteristic impedance, the expression for the current distribution at a given observation point z along the channel was obtained by integrating the contributions of all current sources above it as follows:

$$i(z,t) = \int_z^{H(z,t)} di_s\left(z', t - \frac{z'-z}{c}\right)dz' = \int_z^{H(z,t)} g\left(t - \frac{z'}{v} - \frac{z'-z}{c}\right)e^{-\frac{z'}{\lambda}}dz' \quad (28)$$

where $H(z, t)$ is the apparent height of the return stroke wave front as seen by an observer at height z, which is given by

$$H(z,t) = \frac{vc}{v+c}\left(t + \frac{z}{c}\right) \quad (29)$$

In particular, the current at the channel base can be obtained from (28) letting $z = 0$

$$i(0,t) = \int_0^{H(0,t)} g\left(t - \frac{z'}{v} - \frac{z'}{c}\right)e^{-\frac{z'}{\lambda}}dz' \quad (30)$$

Table 1. $P(z)$ and v^* for Five of the Engineering Return Stroke Models[a]

Model	$P(z)$	v^*
BG	1	∞
TCS	1	$-c$
TL	1	v
MTLL	$1-z/H_{tot}$	v
MTLE	$\mathrm{Exp}(-z/\lambda)$	v

[a]*Rakov and Uman* [1998]. H_{tot} is the total return stroke channel height and λ is the attenuation height constant in the MTLE model.

Combining (26) and (28) we can write

$$e^{-\frac{z}{\lambda}} i\left(0, t-\frac{z}{v}\right) = \int_{z}^{H(z,t)} g\left(t-\frac{z'}{v}-\frac{z'-z}{c}\right) e^{-\frac{z'}{\lambda}} dz' \tag{31}$$

Now, let us include reflections both at ground level and at the return stroke wave front in the analysis. In doing so, the return stroke channel is assumed to be a transmission line whose bottom end is fixed at ground level and features a constant, frequency-independent reflection coefficient, ρ_g, for downward current waves and its upper end is extending with speed v, featuring reflections characterized by the Doppler effect for upward propagating current waves formulated in section 2. Any downward wave, when reflected upward from the channel base, acts a source located at the lower end of the transmission line model of the lightning return stroke channel in a similar way as shown in Figure 1. For the case where $z' > z$ (Figure 2a), the elemental current seen by an observer at z due to a current source at z' can be written as

$$\begin{aligned} di_1(z,z',t) = e^{-\frac{z'}{\lambda}} dz' \Big\{ & g\left(t-\frac{z'}{v}-\frac{z'}{c}+\frac{z}{c}\right) + \rho_g g\left(t-\frac{z'}{v}-\frac{z'}{c}-\frac{z}{c}\right) \\ & - \rho_g g\left(k\left(t-\frac{z'}{v}-\frac{z'}{c}+\frac{z}{c}\right)\right) - \rho_g^2 g\left(k\left(t-\frac{z'}{v}-\frac{z'}{c}-\frac{z}{c}\right)\right) \\ & + \rho_g^2 g\left(k^2\left(t-\frac{z'}{v}-\frac{z'}{c}+\frac{z}{c}\right)\right) + \rho_g^3 g\left(k^2\left(t-\frac{z'}{v}-\frac{z'}{c}-\frac{z}{c}\right)\right) \\ & - \rho_g^3 g\left(k^3\left(t-\frac{z'}{v}-\frac{z'}{c}+\frac{z}{c}\right)\right) + \ldots \Big\} \end{aligned} \tag{32}$$

where k is given by

$$k = \frac{c-v}{c+v} \tag{33}$$

Regrouping similar terms, we can write

$$\begin{aligned} di_1(z,z',t) = e^{-\frac{z'}{\lambda}} dz' \times \Big\{ & g\left(t-\frac{z'}{v}-\frac{z'-z}{c}\right) \\ & + \sum_{n=1}^{\infty} (-1)^{n+1} \rho_g^n g\left(k^{n-1}\left(t-\frac{z'}{v}-\frac{z'-z}{c}-\frac{2z}{c}\right)\right) \\ & + \sum_{n=1}^{\infty} (-1)^{n} \rho_g^n g\left(k^{n}\left(t-\frac{z'}{v}-\frac{z'-z}{c}\right)\right) \Big\} \end{aligned} \tag{34}$$

For the case where $z > z'$ (Figure 2b), the elemental current seen by an observer at z due to a current source at z' can be written as

$$\begin{aligned} di_2(z,z',t) = e^{-\frac{z'}{\lambda}} dz' \Big\{ & \rho_g g\left(t-\frac{z'}{v}-\frac{z'}{c}-\frac{z}{c}\right) \\ & - \rho_g g\left(k\left(t-\frac{z'}{v}-\frac{z'}{c}+\frac{z}{c}\right)\right) - \rho_g^2 g\left(k\left(t-\frac{z'}{v}-\frac{z'}{c}-\frac{z}{c}\right)\right) \\ & + \rho_g^2 g\left(k^2\left(t-\frac{z'}{v}-\frac{z'}{c}+\frac{z}{c}\right)\right) + \rho_g^3 g\left(k^2\left(t-\frac{z'}{v}-\frac{z'}{c}-\frac{z}{c}\right)\right) \\ & - \rho_g^3 g\left(k^3\left(t-\frac{z'}{v}-\frac{z'}{c}+\frac{z}{c}\right)\right) + \ldots \Big\} \end{aligned} \tag{35}$$

Regrouping similar terms, we can write

$$\begin{aligned} di_2(z,z',t) = e^{-\frac{z'}{\lambda}} dz' \Big\{ & \sum_{n=1}^{\infty} (-1)^{n+1} \rho_g^n g\left(k^{n-1}\left(t-\frac{z'}{v}-\frac{z'-z}{c}-\frac{2z}{c}\right)\right) \\ & + \sum_{n=1}^{\infty} (-1)^{n} \rho_g^n g\left(k^{n}\left(t-\frac{z'}{v}-\frac{z'-z}{c}\right)\right) \Big\} \end{aligned} \tag{36}$$

The total current at a height z due to such a distributed current source representation can then be obtained by integrating (34) and (36) as follows:

$$i(z,t) = \int_{0}^{z} di_2(z,z',t) dz' + \int_{z}^{H(z,t)} di_1(z,z',t) dz' \tag{37}$$

Replacing (34) and (36) into (37), we obtain

$$\begin{aligned} i(z,t) = & \int_{z}^{H(z,t)} e^{-\frac{z'}{\lambda}} dz' g\left(t-\frac{z'}{v}-\frac{z'-z}{c}\right) \\ & + \int_{0}^{H(z,t)} e^{-\frac{z'}{\lambda}} dz' \sum_{n=1}^{\infty} (-1)^{n+1} \rho_g^n g\left(k^{n-1}\left(t-\frac{z'}{v}-\frac{z'-z}{c}-\frac{2z}{c}\right)\right) \\ & + \int_{0}^{H(z,t)} e^{-\frac{z'}{\lambda}} dz' \sum_{n=1}^{\infty} (-1)^{n} \rho_g^n g\left(k^{n}\left(t-\frac{z'}{v}-\frac{z'-z}{c}\right)\right) \end{aligned} \tag{38}$$

Finally, using (31), we can simplify (38) to yield

$$\begin{aligned} i(z,t) = & e^{-\frac{z}{\lambda}} i\left(0, t-\frac{z}{v}\right) + \sum_{n=1}^{\infty} (-1)^{n+1} \rho_g^n i\left(0, k^{n-1}\left(t-\frac{2z}{c}\right)\right) \\ & + \sum_{n=1}^{\infty} (-1)^{n} \rho_g^n i(0, k^n t) \end{aligned} \tag{39}$$

Equation (39) is a generalization of the MTLE model in which reflections at ground level and at the return stroke wave front are both taken into account. In generalizing the above derivation to other engineering models, we follow the approach used by *Rachidi et al.* [2002]. In this regard, we note that the following expression can be used to express the spatial-temporal distribution of the current for most of the engineering return stroke models [*Rakov and Uman*, 1998]

$$i(z,t) = P(z) i\left(0, t-\frac{z}{v^*}\right) u\left(t-\frac{z}{v}\right) \tag{40}$$

where $P(z)$ is the attenuation function, v is the return stroke wave front speed defined earlier, and v^* is the current wave speed. $P(z)$ and v^* for five of the engineering return stroke models to be discussed in what follows are shown in Table 1.

[19] Applying the same procedure that led to (39) for the other four models, we arrive at the following general expression for the current wave form along the channel accounting for reflections at ground and at the return stroke wave front taking into account the Doppler effect:

$$i(z,t) = P(z)i\left(0, t - \frac{z}{v^*}\right) + \sum_{n=1}^{\infty}(-1)^{n+1}\rho_g^n i\left(0, k^{n-1}\left(t - \frac{2z}{c}\right)\right) + \sum_{n=1}^{\infty}(-1)^n \rho_g^n i(0, k^n t) \quad (41)$$

4. Discussion

[20] The derived equations (15) and (23) imply that it is not possible to represent reflections occurring at the extending end of a transmission line using a constant, frequency-independent reflection coefficient, as previously done in the lightning literature. The reflected wave from an extending transmission line suffers distortion, the amount of which depends on the incident wave form and its frequency content.

[21] When the speed of the extending transmission line is much smaller than that of the propagating pulses, $v \ll c$, it is easy to see that equations (15) and (23) reduce to the expressions for a classical reflection from a static open-circuited transmission line for which the reflection coefficient is equal to -1. On the other hand, if the speed of the extending transmission line is assumed equal to the speed of the propagating pulses ($v = c$), careful examination of (15) and (23) shows that no reflection would occur at the extending end of the transmission line.

[22] The revised expression for engineering return stroke models (41) accounts rigorously for the boundary condition at the extending return stroke wave front and guarantees therefore the current continuity. Also, the formulation is shown to be consistent with the relativistic Doppler effect, which was not accounted for in previous studies. Indeed, in such a speed range (speeds near the speed of light), any formulation should satisfy the special theory of relativity and the Lorentz transformation.

5. Conclusions

[23] The possibility of current reflections occurring at the extending end of a return stroke channel has been considered in several recent studies and included in the return stroke models assuming a constant reflection coefficient at the return stroke wave front.

[24] In this paper, we presented a thorough analysis of current wave propagation with arbitrary speed along an extending transmission line. We derived rigorous analytical equations in the time and the frequency domains expressing the reflections occurring at the extending end of the line. The derived equations revealed that it is not possible to represent reflections occurring at the extending end of a transmission line using a constant, frequency-independent reflection coefficient, as previously done in the lightning literature. The reflected wave from the extending end of the line was shown to be affected by the Doppler frequency shift. In other words, the reflected wave from an extending transmission line suffers distortion, the amount of which depends on the incident wave form, its frequency content, and the speed of the extending end of the line. The derived expression is found to be in agreement with the relativistic Doppler effect and is consistent with the Lorentz transformation.

[25] Finally, engineering models for return strokes were generalized accounting for reflections at the ground and at the return stroke wave front taking into account the Doppler effect. Closed-form analytical expressions were derived for the spatial-temporal distribution of the current along the channel.

[26] The extension of the presented analysis to include the presence of a tall strike object and an upward connecting leader is straightforward. Work is in progress to analyze the effect of reflections on radiated electromagnetic fields.

[27] **Acknowledgments.** This work is supported by the European COST Action P18 (The Physics of Lightning Flash and Its Effects). Thanks are owed to M. A. Uman, V. A. Rakov, G. Diendorfer, and the anonymous reviewers for their precious suggestions.

References

Bruce, C. E. R., and R. H. Golde (1941), The lightning discharge, *J. Inst. Electr. Eng.*, *88*(6), 487–520.

Cheng, D. K. (1983), *Field and Wave Electromagnetics*, Addison Wesley, Boston, Mass.

Cooray, V. (2003), On the concepts used in return stroke models applied in engineering practice, *IEEE Trans. Electromagn. Compat.*, *45*(1), 101–108, doi:10.1109/TEMC.2002.808041.

Diendorfer, G., and M. A. Uman (1990), An improved return stroke model with specified channel-base current, *J. Geophys. Res.*, *95*(D9), 13,621–13,644, doi:10.1029/JD095iD09p13621.

Heidler, F. (1985), Traveling current source model for LEMP calculation, paper presented at 6th Symposium and Technical Exhibition on Electromagnetic Compatibility, Swiss Fed. Inst. of Technol., Zurich, Switzerland.

Heidler, F., and C. Hopf (1994a), Lightning current and Lightning electromagnetic impulse considering current reflection at the Earth's surface, paper presented at 22nd International Conference on Lightning Protection, Tech. Univ. of Budapest, Budapest, Hungary.

Heidler, F., and C. Hopf (1994b), Influence of the lightning channel termination on the lightning current and lightning electromagnetic impulse, paper presented at International Aerospace and Ground Conference on Lightning and Static Electricity, Bund. Fuer Wehrverwaltung und Wehrtechnik, Mannheim, Germany.

Jackson, J. (1999), *Classical Electrodynamics*, 3rd ed., Wiley, New York.

Janischewskyj, W., V. Shostak, and A. M. Hussein (1998), Comparison of lightning electromagnetic field characteristics of first and subsequent return strokes to a tall tower: 1. Magnetic field, paper presented at 24th International Conference on Lightning Protection, Staffordshire Univ., Birmingham, U.K., Sept.

Mosaddeghi, A., F. Rachidi, M. Rubinstein, F. Napolitano, D. Pavanello, V. Shostak, W. Janischewskyj, and M. Nyffeler (2010), Radiated fields from lightning strikes to tall structures: Effect of reflections at the return stroke wavefront and an upward connecting leader, *IEEE Trans. Electromagn. Compat.*, *53*(2), 437–445.

Napolitano, F., and C. A. Nucci (2009), An engineering model accounting multiple reflections at the wavefront of return strokes to tall towers, in 4th International Workshop on Electromagnetic Radiation from Lightning to Tall Structures, Univ. of Toronto, Montreal, Quebec, Canada.

Nucci, C. A., C. Mazzetti, F. Rachidi, and M. Ianoz (1988), On lightning return stroke models for LEMP calculations, paper presented at 19th International Conference on Lightning Protection, Aust. Electr. Assoc., Graz, Austria, May.

Nucci, C. A., G. Diendorfer, M. Uman, F. Rachidi, M. Ianoz, and C. Mazzetti (1990), Lightning return stroke current models with specified channel-base current: A review and comparison, *J. Geophys. Res.*, *95*(D12), 20,395–20,408, doi:10.1029/JD095iD12p20395.

Rachidi, F., and C. A. Nucci (1990), On the Master, Uman, Lin, Standler and the modified transmission line lightning return stroke current models, *J. Geophys. Res.*, *95*(D12), 20,389–20,393, doi:10.1029/JD095iD12p20389.

Rachidi, F., V. A. Rakov, C. A. Nucci, and J. L. Bermudez (2002), Effect of vertically extended strike object on the distribution of current along the lightning channel, *J. Geophys. Res.*, *107*(D23), 4699, doi:10.1029/2002JD002119.

Rakov, V. A., and A. A. Dulzon (1987), Calculated electromagnetic fields of lightning return strokes, *Tekh. Elektrodin.*, *1*, 87–89.

Rakov, V. A., and F. Rachidi (2009), Overview of recent progress in lightning research and lightning protection, *IEEE Trans. Electromagn. Compat.*, *51*(3), 428–442, doi:10.1109/TEMC.2009.2019267.

Rakov, V. A., and M. A. Uman (1998), Review and evaluation of lightning return stroke models including some aspects of their application, *IEEE Trans. Electromagn. Compat.*, *40*(4), 403–426, doi:10.1109/15.736202.

Raysaha, R. B., U. Kumar, V. Cooray, and R. Thottappillil (2010), Special case of lightning strike to tall objects on ground, paper presented at 30th International Conference on Lightning Protection, Univ. of Cagliari, Cagliari, Italy, 13–17 Sept.

Rubinstein, M., and M. A. Uman (1991), Transient electric and magnetic fields associated with establishing a finite electrostatic dipole, revisited, *IEEE Trans. Electromagn. Compat.*, *33*(4), 312–320, doi:10.1109/15.99112.

Schulz, W., and G. Diendorfer (1995), Properties of an extended Diendorfer Uman return stroke model [lightning], paper presented at Ninth International Symposium on High Voltage Engineering, Inst. of High Voltage Eng., Graz, Austria.

Shostak, V., W. Janischewskyj, A. M. Hussein, J. S. Chang, and B. Kordi (1999), Return-stroke current modeling of lightning striking a tall tower accounting for reflections within the growing channel and for upward-connecting discharges, paper presented at 11th International Conference on Atmospheric Electricity, Global Hydrol. Clim. Cent., Guntersville, Ala.

Shostak, V., W. Janischewskyj, A. Hussein, and B. Kordi (2000), Electromagnetic fields of lightning strikes to a tall tower: A model that accounts for upward-connecting discharges, paper presented at 25th International Conference on Lightning Protection, Univ. of Patras, Rhodes, Greece.

Tesche, F. M., M. Ianoz, and T. Karlsson (1997), *EMC Ananlysis Methods and Computational Models*, Wiley Intersci., New York.

Thottappillil, R., D. K. McLain, M. A. Uman, and G. Diendorfer (1991), Extension of the Diendorfer-Uman lightning return stroke model to the case of a variable upward return stroke speed and a variable downward discharge current speed, *J. Geophys. Res.*, *96*(D9), 17,143–17,150, doi:10.1029/91JD01764.

Thottappillil, R., V. A. Rakov, and M. A. Uman (1997), Distribution of charge along the lightning channel: Relation to remote electric and magnetic fields and to return-stroke models, *J. Geophys. Res.*, *102*(D6), 6987–7006, doi:10.1029/96JD03344.

Uman, M. A., and D. K. McLain (1969), Magnetic field of the lightning return stroke, *J. Geophys. Res.*, *74*(28), 6899–6910, doi:10.1029/JC074i028p06899.

F. Rachidi and A. Shoory, Electromagnetic Compatibility Laboratory, Swiss Federal Institute of Technology, CH-1015 Lausanne, Switzerland. (farhad.rachidi@epfl.ch; abdolhamid.shoory@epfl.ch)

M. Rubinstein, Department of Electrical Engineering and Computer Science, University of Applied Sciences of Western Switzerland, CH-1400 Yverdon-les-Bains, Switzerland. (marcos.rubinstein@heig-vd.ch)