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Computation of portfolio hedging strategies using a reduced Monge-Ampère equation

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Abstract. Fully nonlinear partial differential equations have been used in the literature when deriving models for optimal portfolio hedging under particular market dynamics. More precisely, some models lead to finding the solution of a Monge-Ampère-type equation, together with the Black and Scholes model, when using a hyperbolic absolute risk aversion utility function in portfolio optimization. Following [13,14], we derive several simplified Monge-Ampère equations for the determination for optimal portfolio strategies based on various market stochastics dynamics. This fully nonlinear equation does not necessarily admit an explicit solution so that numerical methods are required to obtain an approximate solution, and numerical techniques must be chosen appropriately. We extend previous works about a computational framework for the elliptic Monge-Ampère equation into two directions: we relax the assumptions on the data to treat stationary examples with singularities, and we address the non-stationary case by introducing an evolutive Monge-Ampère equation. Preliminary numerical results are presented to validate the efficiency and the robustness of the methods in the stationary case, and a novel computational method is proposed for the non-stationary one.

Keywords: Monge-Ampère equation, evolutive Monge-Ampère equation, optimal portfolio, utility function, computational method, portfolio choice, reduced models.

JEL classification codes: C02, C61, C63, G11.

1 Introduction

We address the problem of selecting an optimal portfolio and derive, static or dynamic, optimal investment strategies. In order to find the optimal investment strategy for a portfolio, we have to determine the percentage of the initial wealth that goes into each asset. To that aim, we consider the *value function*, which allows to describe the wealth over time depending on the original amount invested at the initial time.

In most of the models, the behavior of the assets is usually assumed to be governed by stochastic dynamics [2,8–10]. The value function typically depends on the variables of the underlying stochastic dynamics of the market, and represents the value of the portfolio at each time. Several models can be devised by the determination of the value function. We follow here the approach proposed in [13,14] that advocates the transformation of the Hamilton-Jacobi-Bellman equation into a Monge-Ampère equation for the determination of the value function.

The Monge-Ampère equation is a prototypical example of fully nonlinear equations. These equations have been derived when modeling the selection of optimal portfolios under stochastic dynamics for the assets prices, index prices or even momentum variables [12,14]. Fully nonlinear equations have been used in [7,11,15] when considering, e.g., incomplete markets, or singular transactions. In [13,14], a model for portfolio hedging has been presented under various sets of hypotheses and underlying stochastic dynamics. It is shown how the Hamilton-Jacobi-Bellman partial differential equation can be transformed into an associated Monge-Ampère equation that has to be solved numerically. Here, we build on this approach and propose novel computational techniques for simplified problems arising from this model.

The numerical approximation of fully nonlinear equations may be achieved with several numerical methods. Numerical methods are mainly based on viscosity solutions [3,4], and we refer the reader to [5] for a review of several numerical methods. Based on [1], a least-squares approach has been developed to calculate *stationary* solutions of a prototypical Monge-Ampère equation in a least-squares sense, and we apply it here to the canonical Monge-Ampère equation with singularities. In a second part, independent of the first part, we propose an operator splitting strategy to compute *non-stationary* solutions of the evolutive Monge-Ampère equation. The goal of the present work is to derive and develop these computational approaches, and apply them to simple test cases that may be arising in optimal portfolio strategy.

2 Optimal portfolio investment strategies under stochastic dynamics: a first model

We review a first model for the stochastic control problem of optimal portfolio selection. Let us assume an investor can select a risk-free cash account, with value $C(t)$, that is available on the market, and m assets whose prices $S(t)$ are governed by stochastic differential equations. The equations governing the dynamics are

$$\begin{aligned} dC(t) &= rC(t)dt + dc(t), \\ dS(t) &= S(t)\left(adt + \sigma dB(t)\right), \end{aligned}$$

where r is the risk-free interest rate, $dc(t)$ are the (potential) cash transactions (see, e.g., [13]), a is the appreciation rate, σ the volatility matrix, and $B(t)$ a vector of independent Brownian motions [10].

Let us thus consider a portfolio composed by m assets and the cash account. The total value of such a portfolio at each time is denoted by $V(t)$. The portfolio investment strategy corresponds thus to finding a vector-valued function $\mathbf{X}(t) = \{X_1(t), X_2(t), \ldots, X_m(t)\}$, where $X_j(t)$ is the cash value of the investment deposited in the asset j considered for trading (the remaining part $C(t)$ going to the cash account). For a given strategy, the total portfolio value $V^{\mathbf{X}}(t)$ can be written as:

$$V^{\mathbf{X}}(t) = C(t) + \sum_{j=1}^{m} X_j(t). \tag{1}$$

Starting with a total value of investment V at time t (an initial time that can be, for instance, today), the objective of the investor is to maximize the expected value of the utility ψ of the wealth at a given time horizon T with $0 < t < T$. This can be formally written as finding the value $v(t, V)$ such that:

$$v(t, V) = \max_{\mathbf{X}} E_{t,V}\left(\psi\left(V^{\mathbf{X}}(T)\right)\right)$$

where $E_{t,V}(\cdot)$ is the conditional expected value, under the condition that the value of the portfolio at initial time t is V. The choice of the utility function is important as it has implications on the strategy to adopt. Following [14], the *hyperbolic absolute risk aversion* (HARA) family of utility functions is considered. Defined for any parameter $\gamma \in (0, \infty)$, it is given by

$$\begin{aligned} \psi_\gamma(X) &= \frac{1}{\gamma - 1} X^{1-\gamma}, \, \gamma > 0, \, \gamma \neq 1, \\ \psi_1(X) &= \log(X), \qquad \gamma = 1. \end{aligned} \tag{2}$$

We denote by $v_\gamma(t, V)$ the value of the portfolio when using this particular family of utility functions to define the optimal investment strategy. According to [10], if a, σ and r are functions of t only (or even constant, typically for the risk-free rate r), then the final value is the solution of a fully nonlinear partial differential equation of the Monge-Ampère type that reads: find $v_\gamma(t, V)$ such that:

$$-\frac{1}{2}(a(t) - r(t))(\sigma(t)\sigma(t)^T)^{-1}(a(t) - r(t)) \left(\frac{\partial v_\gamma}{\partial V} \right)^2 + V \frac{\partial^2 v_\gamma}{\partial V^2} r(t) \frac{\partial v_\gamma}{\partial V} + \frac{\partial v_\gamma}{\partial t} \frac{\partial^2 v_\gamma}{\partial V^2} = 0 \tag{3}$$

together with the terminal condition $v_\gamma(T, V) = \psi_\gamma(V)$ that expresses that the value at the time horizon is nothing but the chosen value of the utility. Without loss of generality, it is assumed here that the matrix $\sigma(t)\sigma(t)^T$ is invertible. Values for $v_\gamma(t, 0) = 0$ and $v_\gamma(t, V_{\max})$ (where $V_{\max}$ is an arbitrary prescribed value) are reasonably defined. The solution to this equation actually admits an analytical formulation, see [13,14].

3 A second model under new stochastic dynamics

Let us consider, in a second step, a more complicated model for the price dynamics, also extracted from [14]. In addition to the asset prices, let us introduce an index whose value/price is denoted by $Y(t)$, and the momentum $A(t)$. Recall that the momentum is the inertial variable describing the tendency for rising asset prices to keep rising, and for falling prices to continue falling. This effect has been related to the irrational behavior of investors. These new instruments obey the following (coupled) stochastic dynamics:

$$\begin{aligned} dY(t) &= Y(t) \left(A(t)dt + \sigma_y(t, Y(t), A(t)) dB(t) \right), \\ dA(t) &= \frac{1}{Y(t)} \left(\frac{2\pi}{p} \right)^2 (e - Y(t)) dt \end{aligned}$$

for $0 < t < T$, with an initial condition $Y(0) = Y_0$ and $A(0) = A_0$. Here the vector-valued function σ_y is the vector-volatility of the index $Y(t)$, that depends on time, and index and momentum values, e is the price-equilibrium, and p price-dynamics period. Note that, the amplification effect for the dynamics of the momentum comes from the factor $e - Y(t)$: when the index is below the price equilibrium, this factor is negative, and thus drives the evolution of the momentum towards smaller values of the momentum. Reciprocally, when $e - Y(t)$ is positive, the momentum increases as its differential is positive. In this extended case, the trading strategies $\mathbf{X}$ are going to be vector-valued functions of the time t, the wealth V, the index-value Y, and the momentum value A. As before, the objective is, for a given HARA utility function ψ, to maximize the expected value of the utility of the final total wealth. This problem thus leads to find the optimal portfolio strategy $\mathbf{X}(t)$, where $X_j(t)$ is the cash value of the investment in the asset j considered for trading, that satisfies:

$$v(t, V, Y, A) = \sup_{\mathbf{X}} E_{t,V,Y,A} \left(\psi(V^{\mathbf{X}}(T)) \right) \tag{4}$$

Here the expected value is taken as the expected value conditionally given that, at time t (say an initial or present value), the total portfolio value is equal to V, the index value is equal to Y, and the momentum is equal to A. The function $v(t, V, Y, A)$ is called again the value function.

The standard formalism for solving this optimal portfolio strategy problem is to solve the associated Hamilton-Jacobi-Bellman equation characterizing the value function. However, it has been shown in [14] that solving the Hamilton-Jacobi-Bellman equation is equivalent to solving a fully nonlinear equation of the Monge-Ampère type that reads as follows: find the value function $v = v(t, V, Y, A)$ such that:

$$\begin{aligned}
&\frac{\partial^2 v}{\partial V^2}\frac{\partial v}{\partial t} + rV\frac{\partial^2 v}{\partial V^2}\frac{\partial v}{\partial V} + \frac{\partial^2 v}{\partial V^2}\frac{\partial v}{\partial Y}YA + \frac{\partial^2 v}{\partial V^2}\frac{\partial v}{\partial A}\frac{1}{Y}\left(\frac{2\pi}{p}\right)^2(e-Y)\\
&+\frac{1}{2}\frac{\partial^2 v}{\partial V^2}\frac{\partial^2 v}{\partial Y^2}Y^2\sigma_y\sigma_y - \frac{1}{2}\left(\frac{\partial v}{\partial V}\right)^2(a_s-r)(\sigma_s\sigma_s^T)^{-1}(a_s-r)\\
&-\frac{\partial v}{\partial V}\frac{\partial^2 v}{\partial V\partial Y}Y\sigma_y\sigma_s^T(\sigma_s\sigma_s^T)^{-1}(a_s-r) - \frac{1}{2}\left(\frac{\partial^2 v}{\partial V\partial Y}\right)^2 Y^2\sigma_y\sigma_s^T(\sigma_s\sigma_s^T)^{-1}\sigma_s\sigma_y = 0
\end{aligned} \tag{5}$$

for all points (t, V, Y, A) such that $0 < t < T$, $V > 0$, and $Y > 0$, together with the terminal condition $v(T, V, Y, A) = \psi(V)$ that translates the fact that the final value is nothing but the value of the utility function. Moreover, in order to have a mathematically closed problem, we ask again the value function to be concave up or down, and that $V < V_{\max}$, and $Y < Y_{\max}$ with corresponding boundary conditions. Details can be found in [14]. An important remark is that the operator in the Monge-Ampère type equation above is not *elliptic*. That mainly comes from the fact that the evolution of the momentum is such that the dynamics of $A(t)$ are amplified when $e - Y(t)$ changes sign. Here we talk about an *hypo-elliptic* behavior.

The problem (5) for the value function v can be further simplified, by making additional assumptions, more or less restrictive. This is the topic of the following sections describing models for various Monge-Ampère type equations.

4 Stationary Monge-Ampère problem

The generic problem for a canonical elliptic Monge-Ampère equation (on a bounded convex domain of $\mathbb{R}^2$), consists in finding a function $\psi(Y, V)$ such that:

$$\frac{\partial^2 \psi}{\partial Y^2}\frac{\partial^2 \psi}{\partial V^2} - \left(\frac{\partial^2 \psi}{\partial Y \partial V}\right)^2 = F \text{ in } \Omega, \quad \psi = G \text{ on } \partial\Omega. \tag{6}$$

where F and G are given functions. Let us consider here that the return on the asset is the same as the risk free interest rate $r = a_s$, and that $\sigma_s = \sigma_y = \mathbf{I}$ (this last assumption is only technical to simplify the coefficients of the equation, but can be relaxed easily). Moreover, we look for *stationary solutions* such that the solution v is independent of the time variable t. In this case, the previous problem for the value function v could be rewritten:

$$\frac{\partial^2 v}{\partial Y^2}\frac{\partial^2 v}{\partial V^2} - \left(\frac{\partial^2 v}{\partial Y \partial V}\right)^2 = \frac{\partial^2 v}{\partial V^2}\frac{\partial v}{\partial Y}\frac{2A}{Y} + \frac{\partial^2 v}{\partial V^2}\frac{\partial v}{\partial A}\frac{1}{Y}\left(\frac{2\pi}{p}\right)^2(e-Y)\frac{2}{Y^2} \tag{7}$$

When computing the value function v, one can implement an iterative fixed-point approach. Namely, one will start with an initial guess of the value function, say v^0; then we compute the right-hand side of the previous Monge-Ampère equation to calculate the associated right-hand side $F := \frac{\partial^2 v^0}{\partial V^2}\frac{\partial v^0}{\partial Y}\frac{2A}{Y} + \frac{\partial^2 v^0}{\partial V^2}\frac{\partial v^0}{\partial A}\frac{1}{Y}\left(\frac{2\pi}{p}\right)^2(e-Y)\frac{2}{Y^2}$; finally we compute the solution of the canonical Monge-Ampère equation (6), to find the new approximation of the value function, say

v^1, of the value function. We then repeat the procedure until convergence to the final value function.

During that process, several instances of (6) would have to be solved, with very different right-hand sides F. In the next section, we describe an algorithm to solve those problems in a robust way, when F presents some singularities. These singularities may come either from the change of sign due to the term $(e - Y)$, or from the lack of regularity of the derivatives $\frac{\partial^2 v}{\partial V^2}, \frac{\partial v}{\partial Y}, \frac{\partial v}{\partial A}$.

Note that the main difficulty in solving (6) is that the right-hand side does not have a given sign (the characteristic of hypo-ellipticity in this framework). The second difficulty is that (6) does not necessarily have a solution (see, e.g., [3,4]). If we need to solve (6) with a variety of right-hand sides F, one should therefore aim for robustness of the algorithm. A robust method based on least-squares has been derived in [1], but only for $F > 0$ (elliptic case). In the remainder of the article, the least-squares approach for the solution of (6) is briefly presented, and extended to the cases when the right-hand side F is not necessarily strictly positive, but can vanish and presents various types of singularities. In Section 6, we present prototypical examples, as we focus on point singularities or line singularities, which can be seen as the worse case scenarios arising when solving (6). The change of sign of F is not investigated here.

5 Computational algorithm for the stationary Monge-Ampère problem

Among the various methods available for the solution of (6), we advocate the following nonlinear least-squares algorithm: find $(\psi, \mathbf{p}) \in V_g \times \mathbf{Q}_f$ satisfying

$$\frac{1}{2}\int_{\Omega} \left|\mathbf{D}^2\psi - \mathbf{p}\right|^2 d\mathbf{x} \leq \frac{1}{2}\int_{\Omega} \left|\mathbf{D}^2\varphi - \mathbf{q}\right|^2 d\mathbf{x}, \quad \forall(\varphi, \mathbf{q}) \in V_g \times \mathbf{Q}_f. \tag{8}$$

Here the functional spaces and sets in (8) are defined by: $V_g = \left\{\varphi \in H^2(\Omega)\,,\, \varphi = G \text{ on } \partial\Omega\right\}$, and $\mathbf{Q}_f = \left\{\mathbf{q} \in L^2(\Omega)^{2\times 2},\ \mathbf{q} = \mathbf{q}^T\,,\, q_{11}q_{22} - q_{12}q_{21} = f,\ q_{11} > 0,\ q_{22} > 0\right\}$. The underlying goal is to introduce the extra variable $\mathbf{p}$ to take into account the Monge-Ampère equation as a constraint.

Then, two outcomes may occur: either there is a solution to the Monge-Ampère problem, and the algorithm will find a solution such that the value of the objective function is zero, or there is no solution, and (8) produces a solution $(\psi, \mathbf{p})$ such that the least-squares distance is non zero, but minimal. The introduction of this extra variable $\mathbf{p}$ allows the decoupling of the differential operators (acting linearly on the unknown function ψ) from the nonlinearities (acting on the unknown tensor-valued function $\mathbf{p}$).

In order to compute a concave up (or a concave down, with a slight modification) solution of problem (8), we advocate an iterative algorithm, which has been thoroughly described in [1]. The principle is to freeze all variables but one at each sub-step, so that the global problem reduces to a sequence of smaller, easier, problems. Dedicated techniques can then be used for each type of problem. First we initialize ψ^0 satisfying the boundary conditions $\psi^0 = G$ on $\partial\Omega$. Then, for $n \geq 0$, assuming that ψ^n is known, we compute $\mathbf{p}^n, \psi^{n+1/2}$ and ψ^{n+1} as follows:

$$\mathbf{p}^n = \arg\min_{\mathbf{q}\in\mathbf{Q}_f} \int_\Omega \left|\mathbf{D}^2\psi^n - \mathbf{q}\right|^2 d\mathbf{x}, \tag{9}$$

$$\psi^{n+1/2} = \arg\min_{\varphi\in V_g} \int_\Omega \left|\mathbf{D}^2\varphi - \mathbf{p}^n\right|^2 d\mathbf{x}, \tag{10}$$

$$\psi^{n+1} = \psi^n + \omega(\psi^{n+1/2} - \psi^n), \tag{11}$$

with $0 < \omega < 2$ a relaxation parameter whose only intend is to speed up the process.

When ψ^n is frozen, the first step of the iterative algorithm does not involve any derivatives. Thus the minimization problem can be solved *point-wise*, for every $\mathbf{x}$ in the domain Ω. This leads to the solution of a family of finite dimensional minimization problem, that are actually the constrainted minimization of a quadratic function. Several methods could be advocated, for instance involving Newton-based methods or algorithms based on the decomposition in eigenvalues and eigenvectors.

When $\mathbf{p}^n$ is frozen, the second step of the iterative algorithm corresponds to a classical problem from calculus of variations. Written in variational form, the Euler-Lagrange equation corresponding to the minimization problem actually represent a linear variational problem of the biharmonic type. A suitable *conjugate gradient solution* in Hilbert spaces can be used for its solution. Combined with an appropriate mixed finite element approximation, it requires, at each iteration of the conjugate gradient algorithm, the solution of two discrete Poisson problems. Details for both steps can be found, e.g., in [1].

Remark 1. We consider finite element based methods for the approximation of (6). More precisely, we favor here a mixed finite element approximation. With this approach, it is possible to solve (6) using approximations commonly used for the solution of second order elliptic problems; here we use piecewise linear and globally continuous finite elements over a discretization of Ω for example. The use of low order finite elements is justified in order to have the flexibility to consider computational domains with arbitrary (convex) shapes.

6 Numerical experiments

Classical numerical experiments have been presented in [1] to validate the methodology from a computational point of view for the stationary problem. The purpose of the results presented here is to discuss cases where the data is singular, and when the solution is non-stationary. We are thus going to consider non-smooth cases where the assumption $f > 0$ is relaxed and replaced by $f \geq 0$.

The computational code used for these simulations has been entirely developed in Fortran. Post-processing is achieved with the (open-source) software Paraview. Typically, the number of iterations for the iterative algorithm is 100 to guarantee reaching a stationary solution. Each of the two examples below corresponds to the calculation of the value function v, as a function of the initial value V and of the index value Y, for a given choice of terminal condition (associated boundary conditions), which is translated into the right-hand side function.

6.1 Non-Smooth Problems Involving Point Singulairities

Set $\Omega = (0,1)^2$. Let us consider the case for which the solution of the associated problem (6) is the function v defined by $v(Y,V) = \sqrt{Y^2+V^2}$ (a convex function which is not regular if

$(0,0)$ belongs to Ω) Actually this test case involves the Dirac measure at $(0,0)$, $\delta_{(0,0)}$, which corresponds to a *point singularity.* The Dirac measure can be related to singular transactions on the market. The first test problem that we consider is: find $v : \Omega \rightarrow \mathbb{R}$ such that

$$\frac{\partial^2 v}{\partial V^2}\frac{\partial^2 v}{\partial Y^2} - \left(\frac{\partial^2 v}{\partial V \partial Y}\right)^2 = 0 \text{ in } \Omega, \qquad v(Y,V) = \sqrt{Y^2+V^2} \text{ on } \partial\Omega.$$

The unique convex solution of this Monge-Ampère problem is the function $v(Y,V) = \sqrt{Y^2+V^2}$. This choice of right-hand side corresponds to a special case of (6), that is one instance that has to be solved during the fixed-point process described in Section 4. Our methodology being theoretically justified for strictly positive right-hand sides f only (see [1]), we approximate this problem by

$$\frac{\partial^2 v_\varepsilon}{\partial V^2}\frac{\partial^2 v_\varepsilon}{\partial Y^2} - \left(\frac{\partial^2 v_\varepsilon}{\partial V \partial Y}\right)^2 = \varepsilon \text{ in } \Omega, \qquad v_\varepsilon(Y,V) = \sqrt{Y^2+V^2} \text{ on } \partial\Omega,$$

where $\varepsilon > 0$ is a (small) positive number. The discrete algorithm based on the least-squares methodology actually provides, without significant problems, a solution to this Monge-Ampère problem. Figure 1 illustrates the approximation of the solution, denoted by $v_{\varepsilon,h}$, obtained with an unstructured isotropic mesh of size h, together with the error (right) introduced by changing the right-hand side F (from 0 to ε). Empirically, the solution obtained by our least-squares/relaxation method is essentially independent of the value of ε, for ε in the range from $\varepsilon = 10^{-1}$ to $\varepsilon = 10^{-9}$. It also illustrates the level lines of the error $||v_{\varepsilon,h} - v||$, showing that the error is maximal where the approximation of the right-hand side has the worst effect (i.e. near the singularity). The least-squares approach is thus efficient in handling the regularized version of the Monge-Ampère equation.

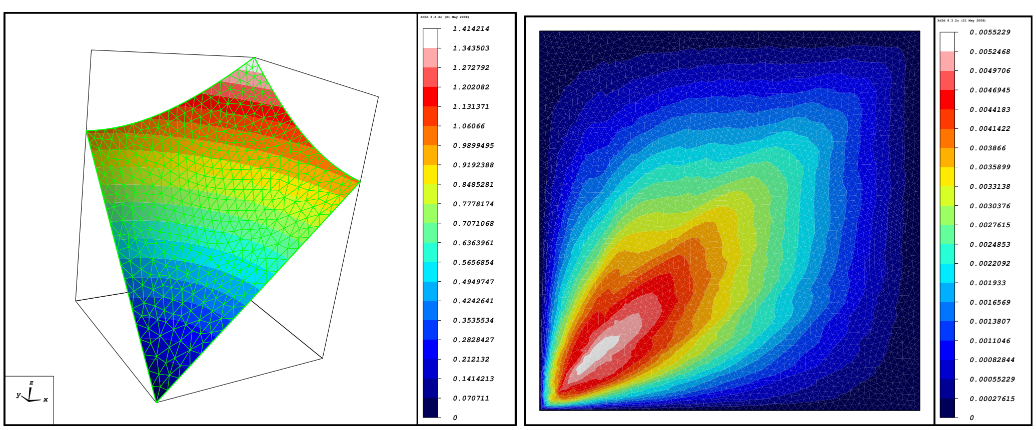

Fig. 1. Non-smooth test problem with a point singularity. Left: Graph of the numerical solution $v_{\varepsilon,h}$ ($h \simeq 0.0509, \varepsilon = 10^{-6}$), and right: level lines of the error $||v_{\varepsilon,h} - v||$.

Next, we consider a related test problem on the unit open disk Ω, that is find the value function v such that

$$\frac{\partial^2 v}{\partial V^2}\frac{\partial^2 v}{\partial Y^2} - \left(\frac{\partial^2 v}{\partial V \partial Y}\right)^2 = \pi \delta_{(0,0)} \text{ in } \Omega, \quad v = 1 \text{ on } \partial\Omega,$$

The unique convex (generalized) solution of this problem is, again, the function v defined by $v(Y,V) = \sqrt{Y^2+V^2}$. From a numerical point of view, the Dirac measure $\delta_{(0,0)}$ is regularized to make it compatible with our least-squares methodology. We thus consider the regularized approximation of the problem ($\varepsilon > 0$):

$$\frac{\partial^2 v_\varepsilon}{\partial V^2}\frac{\partial^2 v_\varepsilon}{\partial Y^2} - \left(\frac{\partial^2 v_\varepsilon}{\partial V \partial Y}\right)^2 = \frac{\varepsilon^2}{(\varepsilon^2 + V^2 + Y^2)^2} \text{ in } \Omega, \qquad v_\varepsilon = 1 \text{ on } \partial\Omega.$$

Figure 2 illustrates the approximation $v_{\varepsilon,h}$ of the solution, together with the level lines of the error, showing the ability of the least-squares approach in handling this regularized problem. Figure 3 illustrates the convergence of the approximation $v_{\varepsilon,h}$ to the exact solution v, for both square and disk domains. Good convergence properties are obtained by appropriately choosing $\varepsilon \simeq h$, h being the mesh size.

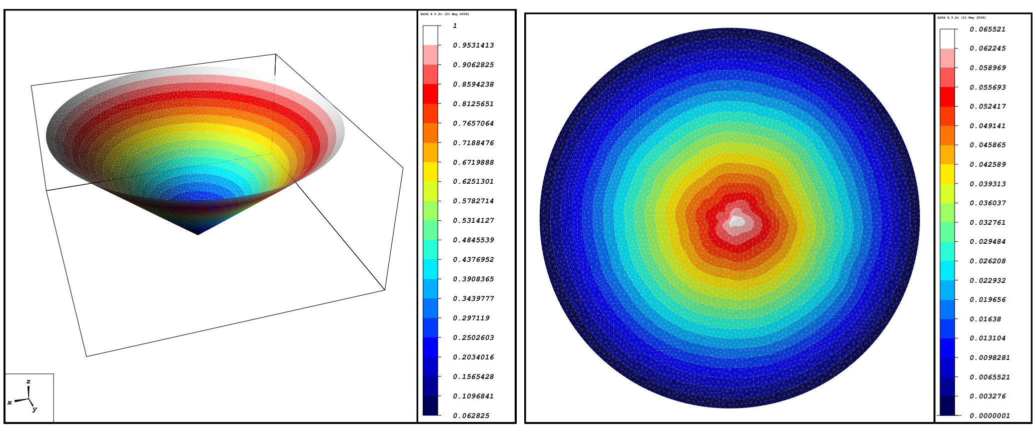

Fig. 2. Graph of the approximated solution $v_{\varepsilon,h}$ on the unit disk ($h \simeq 0.0103, \varepsilon = 10^{-3}$) (left), and level lines of the error $||v_{\varepsilon,h} - v||$.

6.2 Non-Smooth Problems Involving Sharp Edges

A second test case, where the singularity introduced at the right-hand side is now supported by a one-dimensional segment, creating a sharp edge, is now considered. The problem reads as follows: find $v : \Omega \to \mathbb{R}$ such that

$$\begin{cases} \dfrac{\partial^2 v}{\partial V^2}\dfrac{\partial^2 v}{\partial Y^2} - \left(\dfrac{\partial^2 v}{\partial V \partial Y}\right)^2 = 0 \text{ in } \Omega, \\ v(Y,V) = -\min\{Y, 1-Y\} \quad \text{on } \partial\Omega, \end{cases}$$

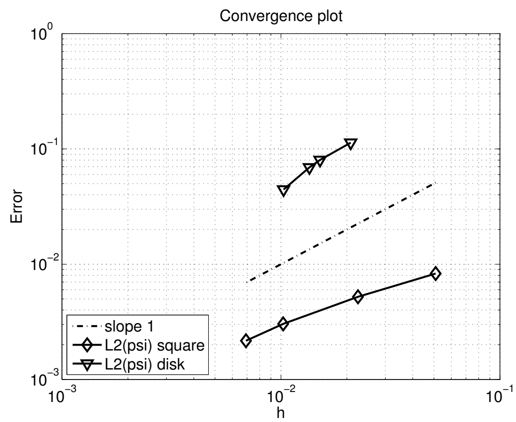

Fig. 3. Convergence of the approximated solution v_h toward the exact solution $v(Y,V) = \sqrt{Y^2+V^2}$ when the mesh size h tends to zero on the unit square and the unit disk.

The unique solution is the function $v(Y,V) = -\min\{Y, 1-Y\}$. Again, the right-hand side is regularized with a suitable expression, namely by introducing a smoothing parameter:

$$\begin{cases} \dfrac{\partial^2 v_\varepsilon}{\partial V^2}\dfrac{\partial^2 v_\varepsilon}{\partial Y^2} - \left(\dfrac{\partial^2 v_\varepsilon}{\partial V \partial Y}\right)^2 = \varepsilon(>0) & \text{in} \quad \Omega, \\ v_\varepsilon(Y,V) = -\min\{Y, 1-Y\} & \text{on} \quad \partial\Omega, \end{cases}$$

Figure 4 illustrates the approximation of the solution $v_{\varepsilon,h}$ obtained with an unstructured isotropic mesh, together with a zoom around the sharp edge. It illustrates the smoothing around the edge. However, the minimal value for the function is numerically equal to -0.4922, instead of -0.5 for the theoretical solution.

7 Evolutive Monge-Ampère problem

In a second step, let us describe another simplified model, for which we do not make the assumption of stationarity. Then, we will write a generic computational method for this simplified model, which is still to be applied to relevant test cases. The generic problem for the *evolutive Monge-Ampère equation* (on a bounded convex domain of $\mathbb{R}^2$) consists in finding $\psi(t,Y,V)$ satisfying

$$\frac{\partial \psi}{\partial t} + K\left(\frac{\partial^2 \psi}{\partial Y^2}\frac{\partial^2 \psi}{\partial V^2} - \left(\frac{\partial^2 \psi}{\partial Y \partial V}\right)^2\right) = F \quad \text{in } \Omega \times (0,T), \tag{12}$$

together with $\psi = G$ on $\partial\Omega \times (0,T)$, and $\psi(0) = \psi_0$ in $\Omega \times \{0\}$, where F, G, K and ψ_0 are given functions. Moreover, we assume that $F, G, K, \psi_0 > 0$.

The problem (5) for the value function v can be addressed again, but while relaxing some of the simplifications introduced in the previous section. We consider here that the return on the asset is the same as the risk free interest rate $r = a_s$, and that $\sigma_s = \sigma_y = \mathbf{I}$ (this last assumption is only technical to simplify the coefficients of the equation, but can be relaxed easily). However,

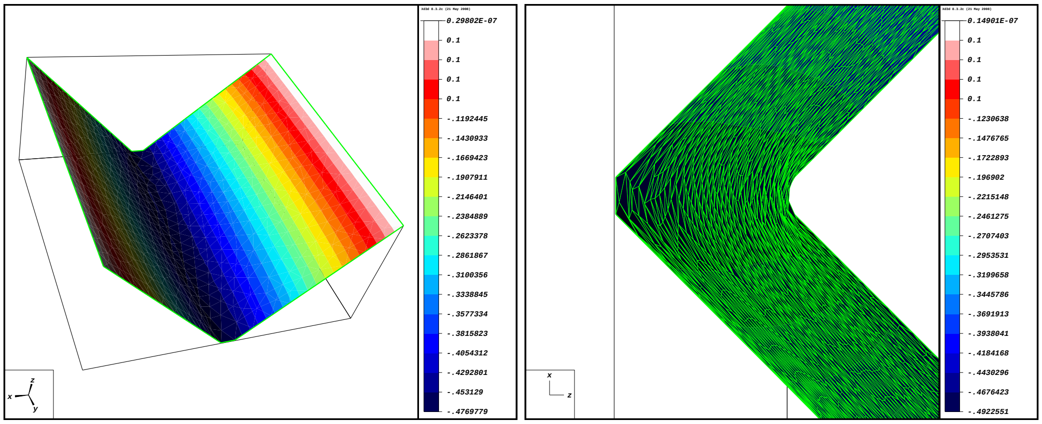

Fig. 4. Non-smooth test problem with a line singularity. Left: Graph of the numerical solution $v_{\varepsilon,h}$ ($h \simeq 0.0509, \varepsilon = 10^{-6}$), and right: zoom around the sharp edge.

we do not look for stationary solutions. In this case, the previous problem for the value function v could be rewritten:

$$\frac{\partial^2 v}{\partial V^2}\frac{\partial v}{\partial t} + \frac{Y^2}{2}\left(\frac{\partial^2 v}{\partial V^2}\frac{\partial^2 v}{\partial Y^2} - \left(\frac{\partial^2 v}{\partial V \partial Y}\right)^2\right) = \frac{\partial^2 v}{\partial V^2}\left(rV\frac{\partial v}{\partial V} + \frac{\partial v}{\partial Y}YA \right.$$
$$\left. + \frac{\partial v}{\partial A}\frac{1}{Y}\left(\frac{2\pi}{p}\right)^2 (e - Y)\right) \tag{13}$$

When computing the value function v, one can again implement an iterative fixed-point approach at each time step. Namely, one will start with an initial guess of the value function, say v^0; then we can solve

$$\frac{\partial^2 v^0}{\partial V^2}\frac{\partial v}{\partial t} + \frac{Y^2}{2}\left(\frac{\partial^2 v}{\partial V^2}\frac{\partial^2 v}{\partial Y^2} - \left(\frac{\partial^2 v}{\partial V \partial Y}\right)^2\right) = \frac{\partial^2 v^0}{\partial V^2}\left(rV\frac{\partial v^0}{\partial V} + \frac{\partial v^0}{\partial Y}YA + \frac{\partial v^0}{\partial A}\frac{1}{Y}\left(\frac{2\pi}{p}\right)^2 (e - Y)\right)$$

Assuming that $\frac{\partial^2 v^0}{\partial V^2} \neq 0$, one can thus define $K := \frac{Y^2}{2} / \frac{\partial^2 v^0}{\partial V^2}$ and $F := rV\frac{\partial v^0}{\partial V} + \frac{\partial v^0}{\partial Y}YA + \frac{\partial v^0}{\partial A}\frac{1}{Y}\left(\frac{2\pi}{p}\right)^2 (e - Y)$. In this case, one has to find a solution to (13) to find the new approximation of the value function, say v^1, of the value function. We then repeat the procedure until convergence to the final value function at each time step.

During that process, several instances of (13) would have to be solved, with very different right-hand sides F and coefficients K. In the next section, we describe briefly a novel algorithm to solve those problems in a robust way, under the assumption that F, K, G and ψ_0 are positive.

8 A novel computational algorithm for the evolutive Monge-Ampère problem

The problem (12) is addressed with a, very classical, explicit, operator splitting approach. Let us define the 2×2 matrices

$$\mathbf{p} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}; \quad \mathbf{D}^2_{\mathbf{x}}\psi = \begin{pmatrix} \dfrac{\partial^2\psi}{\partial V^2} & \dfrac{\partial^2\psi}{\partial V \partial Y} \\ \dfrac{\partial^2\psi}{\partial V \partial Y} & \dfrac{\partial^2\psi}{\partial Y^2} \end{pmatrix}$$

and recall that the determinant of a 2×2 matrix is defined by $\det \mathbf{p} = p_{11}p_{22} - p_{12}p_{21}$. Problem (12) is actually equivalent to finding $(\psi, \mathbf{p})$ such that :

$$\begin{cases} \dfrac{\partial\psi}{\partial t} + K \det \mathbf{p} = F, \\ \mathbf{p} - \mathbf{D}^2_{\mathbf{x}}\psi = 0, \end{cases} \quad \text{in } \Omega \times (0,T).$$

This implies that, by introducing the variable $\mathbf{p}$, we can again decouple effects: the evolutive operator (time derivative) and the constraints on the Hessian $\mathbf{D}^2_{\mathbf{x}}\psi$ (typically that it is symmetric positive definite). Let $\varepsilon > 0$ be a penalization parameter. This problem is transformed into a *flow*, that consist in finding $(\psi, \mathbf{p})$ such that:

$$\begin{cases} \dfrac{\partial\psi}{\partial t} + K \det \mathbf{p} = F, \\ \dfrac{\partial\mathbf{p}}{\partial t} + \dfrac{1}{\varepsilon}\left(\mathbf{p} - \mathbf{D}^2_{\mathbf{x}}\psi\right) = 0, \end{cases} \quad \text{in } \Omega \times (0,T), \tag{14}$$

together with the additional initial condition $\mathbf{p}(0) = \mathbf{D}^2_{\mathbf{x}}\psi_0$. Problem (14) can be solved by an explicit operator splitting method, by freezing alternatively some of the variables and advancing in time. The algorithm stops after a given time, or when a stationary solution is reached. This algorithm is implemented via a time discretization based on N steps and with $\tau = T/N$ the time discretization step. Let us denote by t^n, $n = 0, 1, \dots, N$ the sequence of discrete times. The space discretization is based on a piecewise linear finite element method. It consists of the following three steps:

(Step 1) Find ψ satisfying:

$$\frac{\partial\psi}{\partial t} + K \det \mathbf{p} = F. \tag{15}$$

After explicit discretization, it leads to setting

$$\psi^{n+1} = \psi^n - \tau K \det \mathbf{p}^n + \tau F.$$

(Step 2) Find $\mathbf{p}$ satisfying:

$$\frac{\partial\mathbf{p}}{\partial t} + \frac{1}{\varepsilon}\mathbf{p} = \frac{1}{\varepsilon}\mathbf{D}^2_{\mathbf{x}}\psi \tag{16}$$

After explicit discretization, it leads to setting

$$\mathbf{p}^{n+1/2} = \left(1 + \frac{\tau}{\varepsilon}\right)^{-1} \left(\frac{\tau}{\varepsilon}\mathbf{D}^2_{\mathbf{x}}\psi^{n+1} + \mathbf{p}^n\right).$$

(Step 3) Project $\mathbf{p}$ onto the space of symmetric positive (semi-)definite matrices:

$$\mathbf{p}^{n+1} = \mathbb{P}_+(\mathbf{p}^{n+1/2}). \tag{17}$$

This step is algebraic, and consists in decomposing the matrix $\mathbf{p}^{n+1/2}$ into the SVD decomposition, replace the eigenvalues by their positive parts, and recompose the matrix. This procedure is based on the *polar decomposition*, and is detailed in [6].

9 Conclusions and Perspectives

We have described reduced models for the determination of an optimal portfolio strategy built on the model from [14]. Such reduced models allow to compute the optimal value of the portfolio in various situations, when the assets are governed by stochastic dynamics equations and when the utility function is specifically determined. We have introduced novel computational approaches, based on a least-squares algorithm or an operator splitting strategy, for the solution of simplified Monge-Ampère equations arising in several simplified models. The computational methods are efficient and robust.

Perspectives will include the further investigation of simplified Monge-Ampère equation arising in optimal portfolio investment strategies, by extending the models used here, numerically benchmarking of the evolutive problem, and treating more complicated simulation cases. The next step is to address the robust calculation of non-stationary solutions with evolutive Monge-Ampère equations in various situations.

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