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Extended distribution free newsvendor models with demand updates using experts' judgment

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Abstract

Retailers of short life cycle products, such as [life and style goods are required to commit an order with their suppliers far ahead of their selling seasons inclusive scant demand information. Most of the time, they practice preseason two-stage ordering (instants) that provide an opportunity to modify an initial order based on updated demand forecast obtained at a later stage. The present paper utilizes expert judgment to assess potential impact(s) of contextual information acquired between two instants, in order to revise demand forecast. Additionally, the scant demand information available may not reveal the underlying demand distribution. In this context, we develop inventory models under distribution free newsvendor framework to determine optimal order quantity and weight factor considering also the revised demand forecast. The models consider bidirectional changes in demand and three cases of demand variability: a constant variance case ("CVC"), a constant coefficient of variation case ("CCVC"), and General Case ("GC"). The models developed in the first instance without constraints are subsequently extended by enforcing constraints for practical consideration such as limited storage space or maintenance of pre-defined service level. Moreover, these single-item models are extended to multi-items case to improve their practical utility. The closed form expressions are obtained for decision variables and lower bound of expected profit and their results are discussed using numerical examples. Results show economic benefits in revising the demand forecast using expert judgment and/or negative impact of constraints and/or negative role of demand variability. In addition, a case study is presented to illustrate the potential demand impact assessment and the application of the proposed models within real life circumstances.

Keywords: Single-period inventory; Newsvendor problem; Distribution free demand; Expert judgment; Contextual information, Multi-item inventory models.

1. Introduction

The classical newsvendor problem ("NVP") attempts to determine an order size for a single ordering that maximizes the expected profit under a probabilistic demand framework. NVP reflects many real life situations requiring tradeoff between shortage cost and excess inventory cost. NPV is used in ordering single period products (e.g. fashionable, seasonal and sporting goods) and in managing capacity (e.g. hotel, airline). The basic model and its extensions are extensively discussed in different review papers (Gallego and Moon, 1993; Khouja, 1999; Qin et al., 2011; Kalpana and Kaur, 2012). The NVP is essentially probabilistic and assumes complete knowledge of the underlying probability distribution of the demand while determining the order size. However, in many realistic settings, historical demand data may not fit any standard probability distributions or only partial distribution information is available or reliably estimated. Most of the time, information on demand is no more than a guess of mean and variance with unknown underlying demand distribution . For example, demand information is scarce, imprecise and uncertain owing to the novelty of the product and long procurement lead time. Researchers have assumed demand as being exogenous under different uncertain demand environments (Lau and Lau, 1997a; Choi et al., 2003; Zhou and Wang, 2009; Dutta and Chakraborty, 2010; Yu et al., 2013; Rossi et al., 2014). The distribution free newsvendor problem ("DF NVP") is often used in this case, without making any assumption about the form of the demand distribution. The worst-case expected profit is maximized over the set of distributions satisfying the known information, which is usually the mean and variance of demand. It is referred to as the *distribution free newsboy problem* (Gallego and Moon, 1993). They employ max/min approach that maximizes minimum profit resulting from the worst possible demand distribution.

The single ordering of newsvendor type of products is considered when (i) procurement lead time is longer than the duration of their selling season (ii) a quantity discount scheme offered by the supplier is attractive enough to entice the retailer to make one big order, (iii) the minimum order quantity imposed by the supplier and the retailer's demand is not large enough to support more than one order, and (iv) the fixed ordering setup cost is high (Lau and Lau, 1997b). Retailers of life and style goods (e.g. fashionable apparels) are required to place their orders with suppliers several months before the selling season due to the long procurement lead time and to take advantage of lower prices (Mostard et al., 2011). Moreover, demand information available is scarce because of novelty of the product involved. Thus, these retailers are required to place an order far ahead of the selling season with scarce, uncertain and imprecise demand information. In such circumstances, retailers follow pre-season two stage ordering. It involves placement of a single order before the selling season at two distinct time points (termed as 'stages') satisfying the lead-time requirement (Gurnani and Tang, 1999; Choi et al., 2003). Fig.1 shows a *soft order*, placed at stage zero using initial demand forecast, is allowed to be modified and confirmed using the updated demand forecast as a *confirmed order* at stage one. The selling season is between point 2 and 3 and the full order is received on or before point 2. Unsold leftover inventory, if any, is cleared at salvage value during the clearance season.

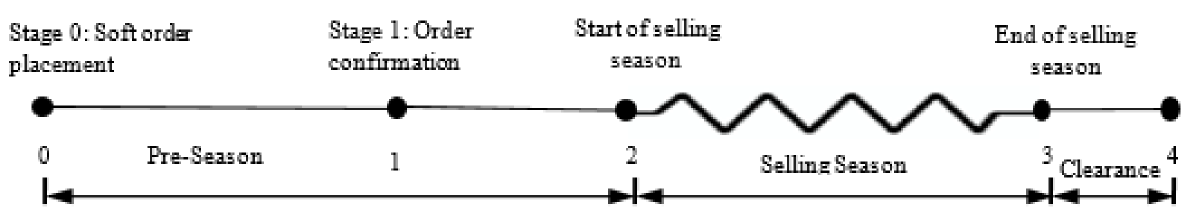

Fig.1. Timeline showing ordering and selling

The demand forecast is revised to incorporate demand impact of additional contextual information (on changing business environment and of events) obtained between the two stages. This demand impact is estimated using experts with domain knowledge and with specific contextual information. Domain knowledge is a knowledge gained by practitioners through experience as a part of their jobs and develops understanding of many cause-effect relationships and environmental cues. Specific information available in the forecast environment is called contextual information and it includes information about events and changes such as unanticipated entry of a competing product, consolidation among competitors, unplanned advertising campaign etc. Domain knowledge enables a practitioner to evaluate the impact of specific contextual information (Webby et al., 2000). The aggregate estimate of potential impacts is termed as the demand forecast adjustment or *demand adjustment* (details are given in Section 3.3, and 6.1). The demand adjustment is integrated with initial demand forecast to determine a revised demand forecast (*revised demand*). Marmier and Cheikhrouhou (2010) presented a factor-based method of estimating and integrating the potential impact with the base demand to determine a revised forecast. The present paper uses their method in obtaining the revised demand with modification. Moreover, we assign a weight to the demand adjustment to signify the degree of acceptance to a decision maker ("DM"), termed as a weight factor $W(0 \leq W \leq 1)$. The demand adjustment would affect both mean and variance of the demand and accordingly different cases of variance changes are modeled (Section 3.3).

The demand adjustment can be positive and negative and therefore both possibilities of demand forecast changes (increase and decrease) required to be considered while modeling. For example, an increase in advertisement expenditure normally leads to an increase in demand. However, in some cases, the net result of advertisement campaign is decrease in demand due to combative countermeasures employed by the competitors and hence both such outcome needs to be considered. Nonetheless, papers like Khouja and Robbins (2003), Lee and Hsu (2011), Dai and Meng (2015) have considered only the unidirectional possibility of demand change versus increase in demand with advertising expenditure. To the best of authors' knowledge, both "positive and/or negative" impact of such unforeseen events through expert judgment under DF NVP setting is not addressed yet. The incorporation of bidirectional demand change warrants two different models - profit maximization for positive adjustment and cost minimization for negative adjustment. However, we formulated a unique objective function that can be used for these two cases (see section 3.4). Thus, the purpose of revising the demand is to integrate demand impact of contextual information obtained between two instants of two-stage ordering with the initial forecast (stage 0) and thereby update the demand forecast. A revised optimal order size and weight factor is determined using the revised /updated forecast and proposed DF NVP. The revised order quantity is used for the placement of a confirm order (stage 1). The revised order size is likely to be more accurate as it considers demand impact estimated by enterprise forecast experts for factors and events happening near to their selling season. With the availability of demand adjustment for contextual information in the context of two- stage ordering, this paper addresses following research questions. (i) How to integrate the demand adjustment with the initial forecast? (ii) What weight should be assigned to the demand adjustment? (iii) How to determine revised order size when only mean and variance is known without knowing underlying demand distribution? (iv) How different cases of variance affect the order size and expected profit? (v) What would order size, weight factor and expected profit when it is subjected constraints for practical considerations? (iv) What would be these decision variables in multi-items case?

In this context, this paper presents inventory models under *distribution free demand NVP* framework to determine an optimal order size and weight factor considering a demand revised for contextual information using expert judgment. These models consider the bidirectional changes in demand and different cases of variance changes. The models developed without constraints in the first instance are subsequently extended by enforcing constraints for the practical considerations of order size (limited budget or storage space) and maintenance of service level defined by a retailer (Jammernegg and Kischka, 2013). Further, these models are extended to multi-items models to make them more realistic. The applicability and effectiveness of the proposed models are presented using numerical examples and a case study.

The paper makes the following important contributions. It is for the first time that demand impact of "soft" contextual information is obtained using expert judgment objectively (following factor-based method) and determines the optimal order size with demand updates. Secondly, proposed models determine optimal order size with scant demand information and without knowing the underlying demand distribution. Thirdly, bi-directional changes in demand (rather than unidirectional) are considered simultaneously through formulation of a unique objective function to determine revised (confirm) order size. Fourthly, multi-items distribution free NVP with and without demand updating is dealt extensively and explicitly. Finally, it demonstrates the applicability and effectiveness of the models developed through a real-life case study.

The rest of the paper is organized as follows. Section 2 provides a relevant literature review. Section 3 formulates inventory models without constraint. Firstly, it presents a summary of basic distribution free NVP followed by model development without constraints for two cases of demand variability and bidirectional demand adjustments. Section 4 extends the models developed in section 3 by enforcing the constraints and subsection 4.3 provides an algorithm to find Lagrange multipliers for the constrained optimization models developed. Section 5 presents multi-items DF NVP with and without revision of demand. Section 6 discusses results using a numerical example. An illustrative case study is presented in Section 7 to demonstrate the applicability of the proposed models. Section 8 concludes the paper and addresses future research directions.

2. Literature review

In this section, the literature related to our work is reviewed and discussed under three categories (i) inventory management with demand-information update, (ii) judgmental adjustment of forecasts, and (iii) distribution free newsvendor problem.

Research in the area of inventory management with demand-information update can be traced back to the 1950s (Dvoretzky et al., 1952). The important research streams include use of time series to update demand forecast, forecast revision and Bayesian methods for updates. The ordering with demand-information updating initiates a field of studies that can be classified as a two-stage inventory decision making problem, where a retailer has two instants for ordering and demand information collected between two instants is used for forecast updates and ordering. Our review mainly restricted to this area. There exist two classes of two-stage inventory newsvendor models (a) Preseason models use exogenous information for updates (b) models that use first period realized demand for updates.

In the first class of models, newsvendor has two instants to order prior to the start of a single selling season and exogenous information collected between the two instants is used to update demand forecast. The early demand signals / commitments, market signals from the sales of other related “pre-seasonal” product can be used to improve the demand forecast. Gurnani and Tang (1999) determine an optimal ordering policy for a newsvendor who has two preseason instants to order. The first sub-period has no demand and exogenous demand information obtained between two instants is used to update the initial forecast for the second period demand. They provide an explicit solution to the cases of worthless and perfect information updates and address a trade-off between improved demand information and potentially increased unit cost at the second instant. Choi et al. (2003) employs a Bayesian approach for the second period demand with some exogenous market information obtained in the first period. Later, Choi et al. (2004) derives an optimal single ordering policy with multiple delivery modes that uses Bayesian methods to update forecast and address a tradeoff between ordering earlier or later. Yan et al. (2003) determines the optimal order size in a two-stage model with dual supply modes and demand information updates. They show that an optimal solution is myopic for uniformly distributed forecast following certain regularity conditions. Ma et al.

(2012) formulates a model in which a retailer has two ordering opportunities prior to realization of demand, considering the forecast updating. Cheaitou (2014) developed a two-period inventory management model where the demand of both periods is stochastic and demand for the second period is updated using exogenous information. Yan and Wang (2014) presented a newsvendor model with capital constraint and demand forecast update with two preseason instants to order from a supplier. Nagare et al. (2016) studies preseason two stage ordering where second stage order is confirmed based on revised demand forecast using enterprise expert judgment.

In the other class of demand forecast updating models, realized demand of the first period is used as endogenous information to update the second period demand. Lau and Lau (1997a) formulated a newsboy model that allows mid-period replenishment after observing an early-season demand to cope with the resource constraint (capacity/budget constraint). It provides semi-analytical solution and analytical conditions for significant profit improvement from reordering. Many other studies employ Bayesian update methods are related to the apparel industry that utilizes the quick response policy (or dual sourcing) to inventory management. They include Iyer and Bergen (1997), Fisher et al. (1994, 2001), Fisher and Raman (1996), Donohue (2000), Gallego and Özer and (2001), Choi et al. (2006), Sethi et al. (2007), Bradford and Sugrue (1990). For instance, Fisher and Raman (1996) modeled the entire sales season demand and the first period demand using a joint probability density function. Iyer and Bergen (1997) analyses the QR system in the fashion industry and uses Bayesian method for updating demand for the second period. They show the decrease in variance of the normally distributed demand. Advanced demand information i.e. early demand signals or commitments is used to improve the forecast of products during the normal selling season (Gallego and Ozer, 2001; Mostard et al., 2011). Sethi et al. (2007) investigate the impact of the forecast quality on the optimal decisions where the buyer can update demand information for the second period and the buyer must commit to a service level (on the basis of market signal). Zheng et al. (2016) investigate the newsvendor model with demand forecast updating under supply constraints. The model allows postponement of the order placement to improve the quality of the demand forecast, while shortening the supply lead time. However, the supplier not only charges a higher cost for shortening lead time, but also sets restrictions on the ordering times and quantities. The model considers dual supply modes: one with a limited ordering time and another one with a decreasing maximum ordering quantity.

In order to increase forecast accuracy and include possible future events that are difficult to address by statistical analysis of historical data, combination of judgmental and mathematical forecasts can provide good results (Webby and O'Connor, 1996, Lawrence et al., 2006). Expert judgment used in conjunction with quantitative methods leads to increased forecast accuracy (Clemen, 1989; Makridakis et al., 2008; Armstrong et al., 2015). In fact, judgmental inputs provide different information, unbiased judgments (Armstrong and Callopy, 1998) and reliable information about some possible future events (Goodwin and Fildes, 1999; Fildes et al., 2009). The adjustment to initial demand forecast is needed to incorporate causal information, extra-model contextual information about past or pending changes and also events affecting the forecast (Flides, 2006; Davydenko and Fildes, 2013). According to Sanders and Ritzman (1995), the revised forecast has better accuracy as experts often have better assumption of potential future events. This type of future events have occurred in the past but are not expected to reoccur in the future, or have not occurred in the past but are expected in the future. Some principles that may be followed for effective use of judgmental adjustment include limiting judgmental adjustments of quantitative forecasts, requiring managers to justify their adjustments in writing, and assessing the results of judgmental interventions (Fildes and Goodwin, 2007). The judgmental adjustment becomes critical with important domain knowledge, high degree of uncertainty, and changes in the demand environment (Sanders and Ritzman, 2001), higher expertise of forecaster, lower credibility of the forecasts system and a strong need of correction (Alvarado-Valencia et al., 2016).

Marmier and Cheikhrouhou (2010) present a factor-based approach to assist the forecaster in focusing selectively on different events and in structuring his judgment when adjusting forecasts. They evaluate the demand impact of contextual information using expert judgment in an objective manner and integrate it into a mathematical forecast. Cheikhrouhou et al. (2011) extend the work to a group of forecasters who provide partial domain knowledge and fragmented contextual information. The approach is based on the identification and the classification of four different types of possible future events. Their impacts are assessed using a fuzzy inference engine that ensures the coherence of the results and limits the biases in decision making. Rekik et al. (2017) analyze judgmental adjustments to replenishment order quantities in a newsvendor setting where information available to managers is reflected in the form of a signal. They find it beneficial even when the probability of a correct signal is not known and offer some interesting insights for judgmentally adjusting order quantities. The present paper uses this method in determining the demand adjustment and more details are given in section 3.3.

Researchers have employed several approaches to deal with the unknown demand distribution and include the Bayesian updates, bootstrapping method, etc. The work of Scarf (1958) on distribution-free ordering is extended by Gallego and Moon (1993) in many directions under a single period framework with distribution free demand using a max-min approach. Moon and Choi (1995) extends the work of Gallego and Moon (1993) to a case where customers may balk if the available inventory level is low. Moon and Silver (2000) develops distribution-free models and heuristics for a multi-item newsboy problem with a budget constraint and fixed ordering costs. Vairaktarakis (2000) develops regret models for multi-item distribution-free NVP under two types of demand uncertainties and a budget constraint. The interval type of demand uncertainty specifies a lower bound and an upper bound on demand, while the discrete type states a set of likely demand values. Alfares and Elmorra (2005) extend Gallego and Moon’s (1993) work with the additional consideration of shortage penalty. Mostard et al. (2005) analyze the distribution-free newsboy problem with returns for a catalogue/internet mail order retailer selling style goods and receiving large numbers of commercial returns. Lee and Hsu (2011) studied effect of advertising on the distribution-free newsboy problem. Liao et al. (2011) extend the newsvendor problem with possible customer balking and a linear lost sales penalty. Zhu et al. (2013) proposed a stochastically robust model for the newsvendor problem where distribution of the random demand is specified only by the mean and either of its standard deviation or its support. Raza (2014) presents a comprehensive analysis of the newsvendor problem with pricing using the distribution-free approach. In today's dynamic market, demand volume, even the underlying demand distribution, changes quickly and the existing methods requiring stationary demand distribution may not work well. In this situation, Zhao et al. (2014) study the non-stationary multi-period newsvendor problem and adopt weal aggregating algorithm to determine the order quantities based on historical demand observations, without knowing the distribution of demand. In this situation, O’Neil et al. (2016) develop a robust and effective machine learning algorithm for newsvendor problems having demand shocks but without having any demand distribution information. Zhang et al. (2017) provide distribution-free methods for the extended multi-period newsboy problems in which the shortage cost and the integral order quantities are considered.

3 DF NVP with demand adjustment using expert judgment

The basic DF NVP is a basis for the development of models with revised demand and hence its summary is provided.

3.1 Notations

P, P_i	unit selling price	Θ	binary variable, 0 for $\Delta_r < 1$ and 1 for $\Delta_r \geq 1$
C, C_i	unit purchase cost	D_0	base demand
S, S_i	unit shortage penalty beyond lost profit	D_1	revised demand forecast
V, V_i	unit salvage value	$\lambda_1, \lambda_2, \lambda_m$	Lagrangian multipliers
C_{Hi}	demand adjustment cost per unit	A, A_i	= (P-C+S) , $(P_i - C_i + S_i)$
Δ, δ	demand adjustment of mean and SD	B, B_i	= (C-V) , $(C_i - V_i)$
Δ_r, δ_r	relative demand adjustment factor		decision variables
α	Pre-defined service level $(0 \leq \alpha < 1)$	Q	order quantity
β	order size constraint parameter $(\beta \geq 0)$	W, W_{1c}, W_{1i}	weight factors
γ, γ_i	exponent of weight factor $(\gamma, \gamma_i > 1)$		

The first and only subscript of 0 and 1 of denote base and revised demands respectively whereas the subscript of i used in multi-item context.

3.2 Basic distribution free NVP

The demand forecast at stage zero is referred to as base demand forecast. Let D_0 be the base demand forecast with known mean μ_0 and variance σ_0^2 but unknown form of its distribution and belongs to the general class G of cumulative distribution functions (CDF). The expected profit for a given order size Q and a base demand D_0, as given in Alfares and Elmorra (2005) is:

$$E(\pi_0) = P\, E[\min(Q, D_0)] - C\, Q + V\, E(Q - D_0)^+ - S\, E(D_0 - Q)^+ \tag{1}$$

Using identities of $\min(Q, D_0) = D_0 - (D_0 - Q)^+$ and $(Q - D_0)^+ = (Q - D_0) + (D_0 - Q)^+$ and the property that- $E(D_0 - Q)^+ \leq \frac{\sqrt{\sigma_0{}^2 + (Q-\mu_0)^2} - (Q-\mu_0)}{2}$, (Gallego and Moon, 1993), a lower bound of expected profit for the worst possible demand scenario can be written as:

$$E(\pi_0^L) = (P - V)\mu_0 - (C - V)Q - (P - V + S)\frac{\sqrt{\sigma_0{}^2 + (Q-\mu_0)^2} - (Q-\mu_0)}{2} \tag{2}$$

The derivatives of $E(\pi_0^L)$ with respect to Q would prove the concavity and provide for the optimal order quantity (Q_0^*) as follows in Eq.3.

$$Q_0^* = \mu_0 + \frac{\sigma_0}{2}\left[\frac{A-B}{\sqrt{A\,B}}\right] \tag{3}$$

The optimal order quantity (Q_0^*) is used to place initial (soft) order at stage 0. The optimal lower bound of the expected profit $E(\pi_0^L)^*$ is given in Eq.4.

$$E(\pi_0^L)^* = (P - C)\mu_0 - \sigma_0\sqrt{A\,B} \tag{4}$$

3.3 Revision of demand using contextual information

The judgmental method employed to improve a mathematical forecast, many a time, is adhoc and on an overall ‘aggregate’ basis. In this context, we employ a factor based approach proposed by Marmier and

Cheikhrouhou (2010) to estimate the aggregate impact of additional demand information, termed as demand adjustment. Experts, such as salesmen or demand forecasters, who have domain knowledge of forecasting and contextual information, are used to identify probable events according to four categories of factors along with their potential impacts, separately. The four factors considered are: quantum jump factor, trend change factor, transient factor and transferred factor. The potential impacts (Δ_j) related to all the factors are summed up to arrive to the aggregate impact, defined as the demand (forecast) adjustment (Δ) i.e. $\Delta = \sum_{j=1}^{4} \Delta_j$. The adjustment to mean of demand can be positive ($\Delta > 0$), negative ($\Delta > 0$) or none ($\Delta = 0$). Similarly, an estimate of adjustment to standard deviation (variance) is obtained and denoted as δ. This method enables a forecaster to communicate more accurately and effectively his implicit knowledge concerning the markets, customers and, competitors by representative factors. The degree of acceptability of this forecast adjustment to a DM may vary in view of their subjective assessment of the operating environment, the competence and credibility of the experts, and the risks involved. Thus, a decision maker may consider only a part of the adjustment for implementation, by assigning a weight W ($0 \leq W \leq 1$) to the demand adjustment, termed as *weight factor*.

The demand adjustment would affect mean and variance/SD of demand. First, the mean of revised demand μ_1 can be defined as $\mu_1 = (\mu_0 + W\Delta)$. To account for the change in demand variance, the demand adjustment is expressed in terms of μ_0 viz. $\Delta = \Delta_r \, \mu_0$ where Δ_r is a relative demand adjustment factor. Thus, we have $\mu_1 = \mu_0(1 + W\Delta_r)$ with $\Delta_r > 0$, $\Delta_r < 0$, or $\Delta_r = 0$ for positive, negative or none demand adjustments. Second, the change in the variance of revised demand considered under three cases: (i) the mean of revised demand changes without altering the variance i.e. $\sigma^2\{D_1\} = \sigma^2\{D_0\} = \sigma_0^2$. This case is referred to as constant variance case (CVC), (ii) the demand adjustment changes both the mean and the variance in a proportion so as the coefficient of variation (CV) remains unchanged i.e. $CV\{D_1\} = CV\{D_0\} = \frac{\sigma_0}{\mu_0}$. Thus, the standard deviation of the revised demand is $\sigma_1 = \sigma_0 + \sigma_0 W\Delta_r = \sigma_0 (1 + W\Delta_r)$. This case is referred to as the constant coefficient of variation case (CCVC) (iii) the adjustments of demand mean and variance can be different in magnitude and direction. Let demand SD (variance) adjustment is δ ($\delta \geq 0$,$\delta < 0$ or $\delta = 0$) and $\delta = \delta_r \, \sigma_0$ and hence revised SD using weight factor would be $\sigma_1 = \sigma_0 + W\delta = \sigma_0 (1 + W\delta_r)$ where $\delta_r < 0$ or $\delta_r = 0$ or $\delta_r > 0$. This case is referred to as General Case (GC) and the earlier two cases are special cases of GC; CVC with $\delta_r = 0$ and CCVC with $\delta_r = \Delta_r$.However, the general case can be $\delta_r \neq \Delta_r$.

Thus, we define D_1 as the revised random demand with known mean (μ_1) and variance (σ_1^2), but with unknown distribution form, belonging to the general class G of cumulative distribution functions. Therefore, the mean and SD of D_1 are as follows:

$$\mu_1 = \mu_0(1 + W\Delta_r) \qquad \Delta_r \geq 0 \text{ or } \Delta_r < 0$$

$$\sigma_1 = \begin{cases} \sigma_0 & \text{for CVC} \\ \sigma_0 (1 + W\Delta_r) & \text{for CCVC} \\ \sigma_0 (1 + W\delta_r) & \text{for GC with } \delta_r < 0 \text{ or } \delta_r \geq 0 \textit{ and } \delta_r \neq \Delta_r \text{ or } \delta_r = \Delta_r \end{cases} \tag{5}$$

3.4 Distribution free NVP with revised demand

Development of the objective function

The expected profit function (1) is modified for the revised demand D_1 and extended to include demand adjustment cost. It has two decision variables: the order quantity (Q) and the weight factor (W), and is given below:

$$E(\pi_1) = P\,E\,[\min(Q\,, D_1)\,] + V\,E(Q - D_1)^+ - C\,Q - S\,E(D_1 - Q)^+ - C_H\mu_0|\Delta_r|W^\gamma \tag{6}$$

Eq. (6) is significantly different from Eq.(1). The replacement of D_0 in (1) by the new demand D_1 in (6) has fundamentally altered the latter because D_1 is an implicit function of the decision variable W, with $\mu_1 = \mu_0(1 + W\Delta_r)$ and different σ_1. Now, Eq. (6) is a bivariate function of Q and W. Secondly, the term; $C_H|\Delta|W^\gamma$ is added to the cost function, represents *demand adjustment cost* required to obtain estimate of demand adjustment and execute a decision based on this adjustment. C_H is a demand adjustment cost per unit [of demand adjusted (Δ)] and can be construed to have many components such as (i) penalty charged per unit by a supplier for modifying an order size (ii) advertising expenditure per unit of accrued demand, or (iii) an adjustment of the objective function per unit revenue loss from discounted sale. Though the demand adjustment cost should vary linearly with W for γ =1. However, the decision variable W will vanish while obtaining its optimal values using derivative with γ =1 and therefore $\gamma \neq 1$. The model requirement is for $\gamma > 1$ and preferred value is 1.4 to 1.8.

Generalization of the model

The maximization of profit function Eq. (6) in the case of $\Delta_r > 0$ would yield $Q^*$and W^*. However, it will not yield the results in the case of $\Delta_r < 0$ as the analytical process of maximizing the profit would always set a decision variable W to zero while attempting to attain greater Q^* by retaining undiminished μ_0 [through $\mu_1 = \mu_0(1 - W\Delta_r)$ considering sign of $\Delta_r < 0$]. Cconsequently, the maximization of the Eq.(6) in the case of $\Delta_r < 0$, would result in ignoring the diminishing impact of the negative demand adjustment. Similarly, the objective of cost minimization is unsuitable for the case of $\Delta_r \geq 0$ as the process of minimization would set the weight factor to zero so as to obtain lower Q^* by retaining μ_0 [through $\mu_1 = \mu_0(1 + W\Delta_r)$]. In other words, an objective of profit (cost) maximization (minimization) would result in ignoring the impact of negative (positive) demand adjustment by setting weight factor to zero, thus losing valuable demand adjustment provided by the experts. Therefore, it is necessary to formulate a single objective function that considers both the demand adjustments and determines the optimal order size and weight factor. This combined function is obtained by inserting a binary variable θ in the expected revenue of the objective function (OF). Using the identities similar to those employed in Eq.(1), we get the expected objective function as follows:

$$E(OF) = (\theta P - V)\mu_1 - (C - V)Q - (P - V + S)\,E(D_1 - Q)^+ - C_H\mu_0|\Delta_r|W^\gamma \tag{7}$$

where $\theta = \begin{cases} 0 & \text{for } \Delta_r < 0 \\ 1 & \text{for } \Delta_r \geq 0 \end{cases}$

The binary variable θ is set to 1 in the case of $\Delta_r \geq 0$ and maximization of $E(OF)$ result in optimal expected profit $[E(\pi_1^*)]$ and determination of W^* and Q^*. The positive demand adjustment improves both the order size and the expected profit and hence it is considered without setting W^* to 0. On the other hand, setting $\theta = 0$ in the case of $\Delta_r < 0$ turns $E(OF)$ into negative valued function and its maximization means minimization of the expected cost. The negative demand adjustment is taken into consideration without setting W^* to 0 as it lowers Q^* and consequently minimizes the expected cost. Thus, the insertion of θ enables the use of Eq.7 in both the cases of demand adjustments.

The substitution of $E(D_1 - Q)^+ = \frac{\sqrt{\sigma_i^2 + (Q-\mu_1)^2} - (Q-\mu_1)}{2}$ in Eq. (7) yields the lower bound of objective function as follows

$$E(OF^L) = (\theta P - V)\mu_1 - (C - V)Q - (P - V + S)\frac{\sqrt{\sigma_i^2 + (Q-\mu_1)^2} - (Q-\mu_1)}{2} - C_H \mu_0 |\Delta_r| W^\gamma \quad (8)$$

The maximization of the lower bound of expected profit in the case of $\Delta_r \geq 0$ is given by

$$\text{Maximize } E(\pi_1^L) = (P - V)\mu_1 - (C - V)Q - (P - V + S)\frac{\sqrt{\sigma_i^2 + (Q-\mu_1)^2} - (Q-\mu_1)}{2} - C_H \mu_0 |\Delta_r| W^\gamma \quad (9)$$

However, in the case of $\Delta_r < 0$, Eq.(8) it is a minimization of the upper bound of expected cost as given by

$$\text{Minimize } E(C0^U) = (C - V)Q + V\mu_1 + (P - V + S)\frac{\sqrt{\sigma_i^2 + (Q-\mu_1)^2} - (Q-\mu_1)}{2} + C_H \mu_0 |\Delta_r| W^\gamma \quad (10)$$

Thus, Eq. (8) considers all the three demand scenarios and represents the general form of the optimization model.

Since the weight factor cannot be more than 1, it becomes necessary to determine the lower bound of C_H that ensures $W \leq 1$. In this regard, we present Lemma1 as follows:

Lemma1. *(i) For given P, C, S, V and γ ; the lower bound of C_H that would ensure optimum weight factor less than or equal to one ($W_1^* \leq 1$) is*

$$C_H \geq \begin{cases} \frac{(\theta P - C)\Delta_r}{|\Delta_r|\gamma} & For\ \ CVC \\ \frac{(\theta P - C)\mu_0\Delta_r - \sigma_0\,\Delta_r\sqrt{A\,B}}{\mu_0|\Delta_r|\gamma} & For\ CCVC \\ \frac{(\theta P - C)\mu_0\Delta_r - \sigma_0.\delta_r.\sqrt{AB}}{|\Delta_r|\ \gamma\ \mu_0} & For\ GC \end{cases}$$

(ii) For given P, C, S, V and γ; any C_H other than that specified in (i) would set $W_1^* = 1$.

The proof of all the Lemmas are given in Appendix B. Considering the Lemma 1, we propose the following Proposition 1.

Proposition 1. *For given P, C, S, V, C_H and γ; the optimal weight factor, order size and lower bound of expected profit for unconstrained DF NVP with revised demand are given as*

(*i*) *Optimal weight factor*

$$W_1^* = \begin{cases} \left[\frac{(\theta P - C)\Delta_r}{C_H|\Delta_r|\gamma}\right]^{\frac{1}{\gamma-1}} & For\ \ CVC \\ \left[\frac{(\theta P - C)\mu_0\Delta_r - \sigma_0\,\Delta_r\sqrt{A\,B}}{C_H\mu_0|\Delta_r|\gamma}\right]^{\frac{1}{\gamma-1}} & For\ CCVC \\ \left[\frac{(\theta P - C)\mu_0\Delta_r - \sigma_0.\delta_r.\sqrt{AB}}{C_H|\Delta_r|.\gamma.\mu_0}\right]^{\frac{1}{\gamma-1}} & For\ GC \end{cases} \quad (11)$$

(ii) Optimal order size (confirmed order)

$$Q_1^* = \mu_1 + \frac{\sigma_1}{2}\left[\frac{(A-B)}{\sqrt{A\,B}}\right] \quad (12)$$

(iii) Optimal expected profit

$$E(\pi_1^L)^* = (P - C)\mu_1 - \sigma_1\sqrt{A\,B} - C_H \mu_0 |\Delta_r| {W_1^*}^{\gamma} \tag{13}$$

Proof of the Propositions 1, 2, 3 and 4 are given in Appendix A, C, D and E respectively.

4 Constrained distribution free NVP with revised demand

The optimal order size based on the revised demand forecast may differ significantly from the initial (base) order. It may be either too large for a given budget or storage space ($\Delta_r > 0$). On the contrary, it may be too small to meet a predefined customer service level ($\Delta_r < 0$). The revised order size is therefore subjected to constraints for the practical considerations.

4.1 Order size constraint

Consider a parameter $\beta\ (\geq 0)$ called *order size constraint parameter*. The revised order size is subjected to an upper limit: $(1+\beta)Q_0^*$. Using Eq. (9), a constrained optimization problem can be stated as determination of the optimal order size and the weight factor that maximizes the lower bound of the expected profit subject to the order size constraint (Abdel-Malek and Montanari 2005).

$$\text{Maximize } E(\pi_{1c}^L) = (P - V)\mu_1 - (C - V)Q - (P - V + S)\frac{\sqrt{\sigma_i^2 + (Q-\mu_1)^2} - (Q - \mu_1)}{2} - C_H \mu_0 |\Delta_r| W^{\gamma}$$

$$\text{Subject to: } Q - (1+\beta)Q_0 \leq 0 \tag{P1}$$

The Lagrange function with $\lambda_2\ (\geq 0)$ multiplier is presented to solve (P1) as follows

$$L = (P - V)\mu_1 - (C - V)Q - (P - V + S)\frac{\sqrt{\sigma_i^2 + (Q-\mu_1)^2} - (Q - \mu_1)}{2} - C_H \mu_0 |\Delta_r| W^{\gamma} - \lambda_1[Q - (1+\beta)Q_0]$$

The $\lambda_1 (\geq 0)$ is a Lagrange multiplier. Here, we present Lemma 2 to determine the lower bound of C_H that ensures $W_{1c}^* \leq 1$.

Lemma 2. *(i) For given P, C, S, V, γ and λ_1; the lower bound of C_H that would ensure constrained optimal weight factor $W_{1c}^* \leq 1$ is*

$$C_H \geq \begin{cases} \dfrac{(P - C - \lambda_1)}{\gamma} & For\ \ CVC \\ \dfrac{(P - C - \lambda_1)\mu_0 - \sigma_0\sqrt{(A - \lambda_1)(B + \lambda_1)}}{\mu_0 \gamma} & For\ CCVC \\ \dfrac{(P - C - \lambda_1)\,\mu_0\,\Delta_r - \sigma_0\ \ \delta_r\sqrt{(A - \lambda_1)(B + \lambda_1)}}{|\Delta_r|\mu_0\gamma} & For\ GC \end{cases}$$

(ii) For given P, C, S, V γ, and λ_1, the C_H other than the specified in (i), would set the constrained optimal weight factor (W_{1c}^*) to 1.

Proposition 2 considers Lemma 2, such as

Proposition 2. *For given P, C, S, V, C_H, γ, and λ_1; the optimal weight factor, the optimal order size and the lower bound of the expected profit for the constrained distribution free NVP using demand revised for contextual information are given as*

(i) Optimal constrained weight factor

$$W_{1c}^{*}=\begin{cases}\left[\frac{(P-C-\lambda_1)}{C_H\gamma}\right]^{\frac{1}{\gamma-1}} & For\ \ CVC\\ \left[\frac{(P-C-\lambda_1)\mu_0-\sigma_0\sqrt{(A-\lambda_1)(B+\lambda_1)}}{C_H\mu_0\gamma}\right]^{\frac{1}{\gamma-1}} & For\ CCVC\\ \left[\frac{(\mathrm{P}-\mathrm{C}-\lambda_1)\ \mu_0\ \Delta_r-\sigma_0\ \ \delta_r\ \sqrt{(A-\lambda_1)(B+\lambda_1)}}{C_H\ |\Delta_r|\mu_0\gamma}\right]^{\frac{1}{\gamma-1}} & For\ GC\end{cases} \tag{14}$$

(ii) Optimal constrained order size

$$Q_{1c}^{*}=\mu_1+\frac{\sigma_1}{2}\left[\frac{(A-\lambda_1)-(B+\lambda_1)}{\sqrt{(A-\lambda_1)(B+\lambda_1)}}\right] \tag{15}$$

(iii) Optimal constrained lower bound of the expected profit

$$E(\pi_{1c}^{L})^{*}=(P-C)\mu_1-\frac{\sigma_1}{2}\left[\frac{B(A-\lambda_1)+A(B+\lambda_1)}{\sqrt{(A-\lambda_1)(B+\lambda_1)}}\right]-C_H\mu_0|\Delta_r|\ (W_{1c}^{*})^{\gamma} \tag{16}$$

4.2 Service level constraint

A service level constraint ensures an adequate inventory to meet pre-defined service level (α) in the face of reduced revised demand forecast i.e. $\Delta_r < 0$. The constraint is imposed in a way that the probability of reaching the service level(α) is higher than η ($0 \le \eta < 1$). The concept is equivalent to the chance constraint discussed in (Charnes and Cooper 1959; Panda et al. 2008; Nagar et al. 2014). It is stated as $\mathrm{Ch}\left(\frac{Q}{D_1} \ge \alpha\right) \ge \eta$; where η is the chance factor. Thus, the constraint is expressed as: $Q \ge \alpha\left(\mu_1+\sigma_i z_\eta\right)$; where $z_\eta = \Phi^{-1}(\eta)$ and Φ is the standard normal cdf. Using Eq. (10), the constrained optimization problem is stated as

$$\text{Minimize } E\left(Co^{U}\right)=(C-V)\,Q+V\,\mu_1+(P-V+S)\frac{\sqrt{\sigma_i^{2}+(Q-\mu_1)^{2}}-(Q-\mu_1)}{2}+C_H\mu_0|\Delta_r|W^{\gamma}$$

$$\text{Subject to: } \quad Q-\alpha\left(\mu_1+\sigma_1 z_\eta\right)\ge 0 \tag{P2}$$

The Lagrange function for (P2) with λ_2 (≥ 0) multiplier is given by

$$L=(C-V)Q+V\mu_1+(P-V+S)\frac{\sqrt{\sigma_i^{2}+(Q-\mu_1)^{2}}-(Q-\mu_1)}{2}+C_H\mu_0|\Delta_r|W^{\gamma}-\lambda_2[Q-\propto(\mu_1+\sigma_i Z_n)]$$

In line with the problem (P1), the Lemma 3 is introduced.

Lemma 3. *(i) For given P, C, S, V, γ and λ_2; a lower bound of C_H that would ensure a constrained optimal weight factor less than or equal to 1 ($W_{1c}^{*} \le 1$) is:*

$$C_H \ge \begin{cases}\frac{[\,C-\lambda_2(1-\alpha)]}{\gamma} & For\ \ CVC\\ \frac{[\,C-\lambda_2(1-\propto)]\mu_0+\sigma_0\left(\sqrt{(A+\lambda_2)(B-\lambda_2)}+\lambda_2\propto Z_n\right)}{\mu_0\cdot\gamma} & For\ CCVC\\ -\frac{[\,C+\lambda(\propto-1)]\mu_0\Delta_r+\sigma_0\delta_r\left(\sqrt{(A+\lambda)(B-\lambda)}+\lambda\propto Z_n\right)}{\mu_0|\Delta_r|\cdot\gamma} & For\ GC\end{cases}$$

(ii) For given P, C, S, V, γ and λ_2, C_H other than the specified in (i) would set the constrained optimum weight factor (W_{1c}^{*}) to 1.

The Proposition 3 is given below in the light of Lemma 3.

Proposition 3. *For given P, C, S, V, γ and λ_2; the optimal constrained weight factor, the optimal order quantity and a lower bound of expected profit are given as*

(i) Optimal constrained weight factor

(ii)
$$W_{1c}^{*} = \begin{cases} \left[\frac{[C-\lambda_2(1-\alpha)]}{C_H.\gamma}\right]^{\frac{1}{\gamma-1}} & For\ CVC \\ \left[\frac{[C-\lambda_2(1-\propto)]\mu_0+\sigma_0\left(\sqrt{(A+\lambda_2)(B-\lambda_2)}+\lambda_2\propto Z_n\right)}{C_H\mu_0.\gamma}\right]^{\frac{1}{\gamma-1}} & For\ CCVC \\ \left[-\frac{[C+\lambda(\propto-1)]\mu_0\Delta_r+\sigma_0\delta_r\left(\sqrt{(A+\lambda)(B-\lambda)}+\lambda\propto Z_n\right)}{C_H\mu_0|\Delta_r|.\gamma}\right]^{\frac{1}{\gamma-1}} & For\ GC \end{cases} \quad (17)$$

(iii) Optimal constrained order size

$$Q_{1c}^{*} = \mu_1 + \frac{\sigma_1}{2}.\left[\frac{(A+\lambda_2)-(B-\lambda_2)}{\sqrt{(A+\lambda_2)(B-\lambda_2)}}\right] \quad (18)$$

(iv) Optimal lower bound of constrained expected profit

$$E(\pi_{1c}^{L})^{*} = (P-C)\mu_1 - \frac{\sigma_1}{2}\left[\frac{B(A+\lambda_2)+A(B-\lambda_2)}{\sqrt{(A+\lambda_2)(B-\lambda_2)}}\right] - C_H\mu_0|\Delta_r|\, {W_{1c}^{*}}^{\gamma} \quad (19)$$

4.3. Algorithm for the optimal Lagrange multipliers

Given the model parameters, it is necessary to find the optimal value of Lagrange multiplier [λ_j with j= 1, 2] for the given constrained optimization problems (P1) and (P2) respectively to determine W_{1c}^{*} and consequently Q_{1c}^{*} and $E(\pi_{1c}^{L})^{*}$

First, we define a function $h(\lambda_j)$, j= 1, 2 such as

$$h(\lambda_j) = \begin{cases} (1+\beta)\,Q_0^{*} - Q & \text{for } \Delta_r \geq 0 \text{ and } j = 1 \\ Q - \alpha\left(\mu_1 + \sigma_1 z_\eta\right) & \text{for } \Delta_r < 0 \text{ and } j = 2 \end{cases}$$

The function $h(\lambda_j)$ is defined in a way that $h(\lambda_j) \geq 0$ would satisfy the constraint for both situations of demand adjustment.

Lemma 4. *There exists a unique* λ_j *with j= 1, 2 such that* $h(\lambda_j) = 0$.

Therefore, a proposed one-dimensional search algorithm would determines λ_j^{*} that satisfies $h(\lambda_j^{*}) = 0$. We designate different values of λ_j with different superscript such as $\lambda_j^0, \lambda_j^1, \lambda_j^2, \ldots . \lambda_j^m$ where m is index and m= 0,1,2, ...The algorithm has the following steps:

Step1. Set $\lambda_j^0 = 0$ and obtain the decision variables W_{1c}^{*} and Q_{1c}^{*}. If the solution satisfies the constraint $h(\lambda_j^0) \geq 0$, the current solution is optimal. Otherwise, it is infeasible and go to step 2.

Step2. As $h(\lambda_j^0) < 0$, set positive value of λ_j viz. λ_j^1 to a large value such that $h(\lambda_j^1) > 0$. Find the λ_j^2 which is the arithmetic mean of λ_j^0 and λ_j^1 viz. $\lambda_j^2 = \frac{\lambda_j^0+\lambda_j^1}{2}$. If h $(\lambda_j^2) \approx 0$, then stop the iterations. Otherwise, proceed as follows:

Select either of λ_j^0 or λ_j^1 such that it will have opposite sign to h (λ_j^2) and find the mean with λ_j^2 to obtain λ_j^3 [viz.$\lambda_j^3 = \frac{\lambda_j^0+\lambda_j^2}{2}$ or $\lambda_j^3 = \frac{\lambda_j^1+\lambda_j^2}{2}$]. Continue the process till it gets h $(\lambda_j^m) = 0$ for m ≥ 3. λ_j^m is the optimal one.

Step3. Use λ_j^* to find W_{1c}^* and whenever $W_{1c}^* > 1$, it is set to 1. Subsequently, determine Q_{1c}^* and $E(\pi_{1c}^L)^*$ using Proposition 2 or Proposition 3.

5. Multi-item DF NVP

5.1 Basic Model

In this section, we present multi-item problem in the presence of a budget constraint while purchasing style goods where a retailer needs to allocate budget among competing items. We denote these items with a index i = 1,2, ..N and the parameters employed earlier are used with this index. The total budget available is denoted by G. In this context, problem is to determine order quantities that maximize the expected profit against the worst possible distribution of the demand without exceeding the budget constraint.

We obtain expression for the lower bound of expected profit for the i^{th} item from (2) as follows

$$E(\pi_{0i}^L) = (P_i - V_i)\,\mu_{0i} - B_iQ_i - \frac{(A_i+B_i)}{2}\left(\sqrt{\sigma_{0i}^2 + Z_i^2} - Z_i\right)$$

Where $A_i = (P_i - C_i + S_i)$, $B_i = (C_i - V_i)$, $(A_i + B_i) = (P_i - V_i + S_i)$ and $Z_i = Q_i - \mu_{0i}$

The problem can be formulated as follows

$$\text{Max } E(\pi_m^L) = \text{Max} \sum_{i=1}^{N} (P_i - V_i)\,\mu_{0i} - B_iQ_i - \frac{(A_i+B_i)}{2}\left(\sqrt{\sigma_{1i}^2 + Z_i^2} - Z_i\right)$$

Subject to: $\sum_{i=1}^{N}[C_iQ_i \le G]$ (P3)

To obtain constrained order size, the Lagrange function formulated with multiplier λ_m (≥ 0) as follows

$$L = \sum_{i=1}^{N}(P_i - V_i)\mu_{0i} - B_iQ_i - \left(\frac{A_i+B_i}{2}\right)\left(\sqrt{{\sigma_{0i}}^2 + {Z_i}^2} - Z_i\right) - \lambda_m\left[\sum_{i=1}^{N} C_iQ_i - G\right]$$

By using $\frac{\partial L}{\partial Q_i} = 0$ for all i, we get optimal constrained order size

$$Q_{0i}^* = \mu_{0i} + \sigma_{0i}\left[\frac{(A_i-\lambda_m C_i)-(B_i+\lambda_m C_i)}{\sqrt{(A_i-\lambda_m C_i)(B_i+\lambda_m C_i)}}\right] \quad (20)$$

The smallest non-negative λ_m that satisfies (P3) can be found using line search algorithm or MS Excel. The expected profit from the i^{th} item is

$$E\left(\pi_{0i}^L\right)^* = (P_i - C_i)\mu_{0i} - \frac{\sigma_{0i}}{2}\left[\frac{B_i(A_i-\lambda_m C_i)+A_i(B_i+\lambda_m C_i)}{\sqrt{(A_i-\lambda_m C_i)\times(B_i+\lambda_m C_i)}}\right] \quad (21)$$

The method used to obtain above expressions are similar to those of (P4) and given in Appendix E. Therefore, total optimal constrained expected profit from all the items

$$E\left(\pi_0^L\right)^* = \sum_{i=1}^{N} E\left(\pi_{0i}^L\right)^* \quad (22)$$

5.2 Model with revised demand

The model considers demand revised for contextual information for an i^th item with revised mean $\mu_{1i} = \mu_{0i}\,(1 + W_{1i}\,\Delta_{ri})$ and standard deviation $\sigma_{1i} = \sigma_{0i}\,(1 + W_{1i}\,\delta_{ri})$. The expected profit from the i^th item is given below. With many items, there would be items with positive, negative or zero demand adjustments and hence this model covers all these possibilities. A lower bound of objective function from the i^th item obtained from (8) is

$$E\left(OF_{1i}^{L}\right) = (\theta P_i - V_i)\,\mu_{1i} - B_i Q_i - \frac{(A_i+B_i)}{2}\left(\sqrt{\sigma_{1i}^2 + Z_{1i}^2} - Z_{1i}\right) - C_{Hi}\mu_{0i}|\Delta_{ri}|W_{1i}^{\gamma_i} \tag{23}$$

where $Z_{1i} = (Q_i - \mu_{1i})$

The multi-item problem is formulated as follows

$$\text{Maximize } E\left(OF_{1}^{L}\right) = \sum_{i=1}^{N}(\theta P_i - V_i)\mu_{1i} - B_i Q_i - \frac{(A_i+B_i)}{2}\left(\sqrt{\sigma_{1i}^2 + Z_{1i}^2} - Z_{1i}\right) - C_{Hi}\mu_{0i}|\Delta_{ri}|W_{1i}^{\gamma_i}$$

$$\text{Subject to: } \sum_{i=1}^{n} C_i Q_i \leq G \tag{P4}$$

The Lagrange function for (P4) is

$$L = \sum_{i=1}^{N}(\theta P_i - V_i)\mu_{1i} - B_i Q_i - \frac{(A_i+B_i)}{2}\left(\sqrt{\sigma_{1i}^2 + Z_{1i}^2} - Z_{1i}\right) - C_{Hi}\mu_{0i}|\Delta_{ri}|W_{1i}^{\gamma_i} - \lambda_m\left[\sum_{i=1}^{N} C_i Q_i - G\right]$$

The Lemma 4 is similar to Lemma 3 and hence avoided. The Proposition 4 is offered in light of Lemma 4.

Proposition 4. For given P_i C_i, S_i, $C_{Hi,}$ V_i, γ_i and λ_m; the *optimal weight factor, the optimal order size and the lower bound of the expected profit for the constrained distribution free NVP using demand revised for contextual information are given as*

(i) Optimal constrained weight factor for the i^th item

$$W_{1i}^{*} = \begin{cases} \left[\dfrac{\theta P_i - C_i\,(1+\lambda_m)}{C_{Hi}\,\gamma_i}\right]^{\frac{1}{\gamma_i - 1}} & \text{For } CVC \\ \left\{\dfrac{[\theta P_i - C_i(1+\lambda_m)]\mu_{0i} - \sigma_{0i}\sqrt{(A_i - \lambda_m C_i)(B_i + \lambda_m C_i)}}{C_{Hi}\mu_{0i}\,\gamma_i}\right\}^{\frac{1}{\gamma_i - 1}} & \text{For } CCVC \\ \left\{\dfrac{[\theta P_i - C_i(1+\lambda_m)]\mu_{0i}\,\Delta_{ri} - \sigma_{0i}\,\delta_{ri}\sqrt{(A_i - \lambda_m C_i)(B_i + \lambda_m C_i)}}{C_{Hi}\mu_{0i}|\Delta_{ri}|\,\gamma_i}\right\}^{\frac{1}{\gamma_i - 1}} & \text{For } GC \end{cases} \tag{24}$$

(ii) Optimal constrained order size for the i^th item

$$Q_{1i}^{*} = \mu_{1i} + \frac{\sigma_{1i}}{2}\left[\frac{(A_i - \lambda_m C_i) - (B_i + \lambda_m C_i)}{\sqrt{(A_i - \lambda_m C_i)(B_i + \lambda_m C_i)}}\right] \tag{25}$$

(iii) Optimal constrained lower bound of the expected profit for i^th item

$$E\left(\pi_{1i}^{L}\right)^{*} = (P_i - C_i)\mu_{1i} - \frac{\sigma_{1i}}{2}\left[\frac{B_i(A_i - \lambda_m C_i) + A_i(B_i + \lambda_m C_i)}{\sqrt{(A_i - \lambda_m C_i)\times(B_i + \lambda_m C_i)}}\right] - C_{Hi}\mu_{0i}|\Delta_{ri}|W_{1i}^{\gamma_i} \tag{26}$$

Lower bound of total expected profit

$$E(\pi_1^{L})^{*} = \sum_{i=1}^{N} E\left(\pi_{1i}^{L}\right)^{*}$$

6 Numerical example

The results of the proposed models are illustrated with numerical examples. The economic parameters are: P = \$35, C = \$20, V = \$12 and S = \$5. The base demand D_0 is assumed to have mean (μ_0) 1000 and standard deviation (σ_0) 200. Optimal order size (Q_0^*) and lower bound of expected profit $\left[E(\pi_0^L)^*\right]$ for the base demand case are determined using (3) and (4) and are 1095 and \$12470 respectively.

6.1 Results with positive demand adjustment

An illustrative potential impact assessment of contextual information using a factor based method is presented in Table 1.

Table 1. Impact assessment of contextual information

Contextual factors		Event	Impact provided by experts
Quantum jump	Δ_1	Addition of new retail outlet	100
Trend change	Δ_2	Increase in product price due to increased input cost	-150
Transient	Δ_3	Marketing and advertisement campaign	300
Transferred impact	Δ_4	Not applicable	--
Demand adjustment $\Delta = \sum_1^4 \Delta_j$			+250

It gives a demand adjustment (Δ) of +250 and relative demand adjustment factor (Δ_r) of 0.25. The results for different C_H (0, 10, 15); γ (1.4, 1.6, 1.8) and β=0.15 are presented in Table 2. It shows that the optimal order size and expected profit with revised demand are larger than those obtained with the base demand. Secondly, the order size constraint places a cap on order size, Q_β [$= (1+\beta)Q_0^*$=1259] and whenever the unconstrained order quantity (Q_1^*) exceed the limit, it is pruned to Q_β by reducing the optimal weight factor. For example, unconstrained CVC with C_H =10, γ =1.6 provides Q_1^*=1319, $E(\pi_1^L)^*$=\$13797 and W_1^*=0.9 whereas with the constraint are Q_{1c}^*=1259, $E(\pi_{1c}^L)^*$= \$13691 and W_{1c}^*= 0.76.

Table 2. Results of positive demand adjustment

		CVC							CCVC						
		Without constraint			Constrained (Order size)				Without constraint			Constrained (Order size)			
C_H	γ	W_1^*	Q_1^*	$E(\pi_1^L)^*$	λ_1^*	W_{1c}^*	Q_{1c}^*	$E(\pi_{1c}^L)^*$	W_1^*	Q_1^*	$E(\pi_1^L)^*$	λ_1^*	W_{1c}^*	Q_{1c}^*	$E(\pi_{1c}^L)^*$
0	1.4	1	1345	16220	5.37	1	1259	16001	1	1369	15587	5.5	1	1259	15302
	1.6	1	1345	16220	5.37	1	1259	16001	1	1369	15587	5.5	1	1259	15302
	1.8	1	1345	16220	5.37	1	1259	16001	1	1369	15587	5.5	1	1259	15302
10	1.4	1	1345	13720	2.15	0.81	1259	13606	0.75	1300	13137	0.63	0.65	1259	13125
	1.6	0.9	1319	13733	1.43	0.76	1259	13691	0.66	1276	13242	0.34	0.63	1259	13239
	1.8	0.8	1294	13797	1.00	0.73	1259	13780	0.63	1268	13346	0.34	0.62	1259	13345
15	1.4	0.43	1203	12932	0	0.43	1202	12932	0.27	1169	12712	0	0.27	1169	12712
	1.6	0.46	1209	13113	0	0.46	1209	13113	0.34	1187	12863	0	0.34	1187	12863
	1.8	0.48	1215	13270	0	0.48	1215	13270	0.38	1199	12998	0	0.38	1199	12998

Thirdly, C_H influences the order size and expected profit mainly through the weigh factor. The C_H less than a certain value, termed as the *threshold demand adjustment cost per unit* (say, C_{HT}), results in $W > 1$ which is subsequently set to 1.The C_{HT} is \$9.4 for unconstrained CVC with γ =1.6. Thus, the optimal order size remains unchanged despite increase in C_H for $C_H \leq C_{HT}$ as W=1; nonetheless the expected profit decreases due to increased demand adjustment cost. However, an increase in C_H beyond C_{HT} results in a decrease in both the order size and expected profit by lowering the weight factor. For instance, increasing C_H from 10 to 20 with γ=1.6 for unconstrained CVC has reduced the order size from 1371 to 1217(11.23%) and expected profit from \$13113 to \$12248 (6.5%) by lowering the weight factor from 0.9 to 0.28. This behaviour is shown in Fig.2. Fourthly, CCVC embodies more demand variability than the CVC for $W^* = 1$ and hence has larger order size, but smaller expected profit than the CVC. On other hand CVC has both larger order size and expected profit than the CCVC for $W^* \neq 1$.

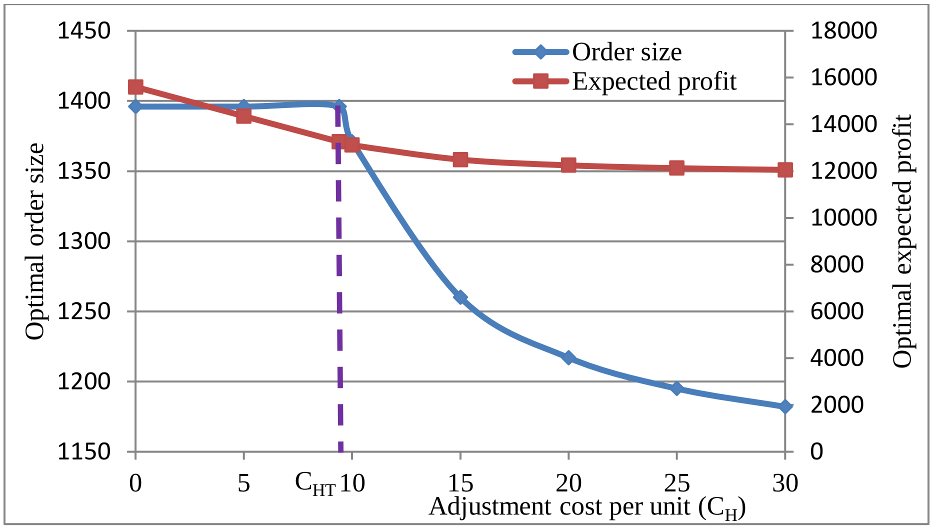

Fig.2. Behaviour of Q^* and $E^L(\pi)^*$ with respect to C_H

5.2 Results with negative demand adjustment ($\Delta < 0$)

A case of negative demand adjustment with $\Delta = -250$ $(\text{i.e.}\, \Delta_r = -0.25)$ is studied retaining the same set parameters. Additionally, α and η are assumed as 0.95 and the results are presented in Table 3.

It shows that negative demand adjustment results in reduced optimal order size and expected profit as compared to those for the base demand. The maximal order size without constraint is 923 for CVC with C_H=15 and γ=1.8 whereas maximal expected profit is \$9352 for unconstrained CCVC with C_H=0. The reduced revised demand forecast has lowered both the order size and expected profit as compared to those obtained in the base demand case viz.Q_0^*=1095 and $E(\pi_0^L)^*$=\$12470. The impact of negative demand adjustment is greatest at optimal weight factor 1. Secondly, it may be noted that the imposition of constraint has reduced the expected profit despite raising the optimal order size. Thirdly, CCVC has lower demand variability than the CVC for W^*=1 and provides for greater expected profit with smaller order size than the CVC. However, in the case of $W^* < 1$, the weight factor is greater for CCVC (hence lower $\mu_1$and σ_1) than in CVC, which results in a smaller order size and lower expected profit in CCVC than in CVC.

Table 3. Results for constrained and unconstrained optimization for negative demand adjustment

		CVC							CCVC						
		Without constraint			Constrained (Service level)				Without constraint			Constrained (Service level)			
C_H	γ	W_1^*	Q_1^*	$E(\pi_1^L)^*$	λ_2^*	W_{1c}^*	Q_{1c}^*	$E(\pi_{1c}^L)^*$	W_1^*	Q_1^*	$E(\pi_1^L)^*$	λ_2^*	W_{1c}^*	Q_{1c}^*	$E(\pi_{1c}^L)^*$
0	1.4	1	845	8720	5.32	1	1025	8140	1	821	9352	5.13	1.00	947	8968

	1.6	1	845	8720	5.32	1	1025	8140	1	821	9352	5.13	1.00	947	8968
	1.8	1	845	8720	5.32	1	1025	8140	1	821	9352	5.13	1.00	947	8968
10	1.4	1	845	6220	5.32	1	1025	5640	1	821	6852	5.13	1.00	947	6468
	1.6	1	845	6220	5.32	1	1025	5640	1	821	6852	5.13	1.00	947	6468
	1.8	1	845	6220	5.32	1	1025	5640	1	821	6852	5.13	1.00	947	6468
15	1.4	0.89	874	5989	5.30	0.86	1059	5670	1	821	5602	5.13	1.00	947	5218
	1.6	0.74	910	7397	5.27	0.73	1091	6976	0.90	849	6496	5.13	0.93	967	5798
	1.8	0.69	923	7984	5.27	0.68	1102	7521	0.80	877	7488	5.13	0.82	1003	6879

5.3 Effect of order size constraint

The impact of β is studied for CVC, Δ_r=0.25, C_H=10, $\gamma = 1.6$ and presented in Fig.3. It can be seen that both Q_{1c}^* and $E(\pi_{1c}^L)^*$ increases with β up to its *limiting value* (β_l) which is 0.21 in this case. The constraint is binding and both Q_{1c}^* and $E(\pi_{1c}^L)^*$ increases with β for $\beta \leq \beta_l$. However, at $\beta = \beta_l$, order size and expected profit value converges to those obtainable in the unconstrained case viz., Q_1^*=1319 and $E(\pi_1^L)^* =$ \$13733.The behaviour of Q_c^* and $E(\pi_{1c}^L)^*$ with respect to β is shown in Fig.3.

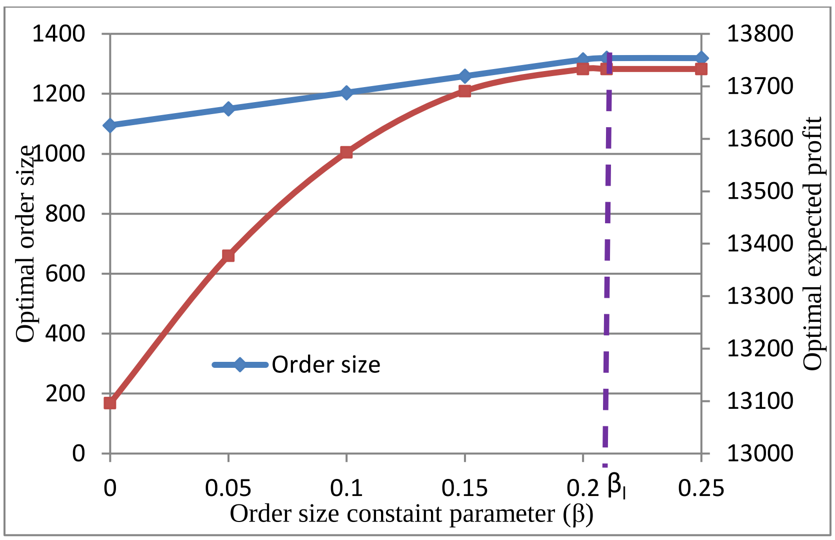

Fig.3. Impact β on Q_c^* and $E^L(\pi_c)^*$

5.4 Effect of service level

The effect of α for Δ_r= - 0.25, C_H=15, $\gamma = 1.6$ and CVC is studied and shown in Fig.4. The constraint becomes binding above a certain value of α, termed as *limiting value of target service level* (α_l) and α_l =0.79 in this case. The constraint is binding in the interval $\alpha_l < \alpha <1$ and Q_{1c}^* increases monotonically with respect to α in this interval. However, $E(\pi_{1c}^L)^*$ first increases respect to α till the latter reaches an optimal value, α^* and thereafter decreases. The optimal value of the service level α^* is 0.84 and can be interpreted that given the economic parameters, service level of 84% provides the optimal expected profit of \$7526.

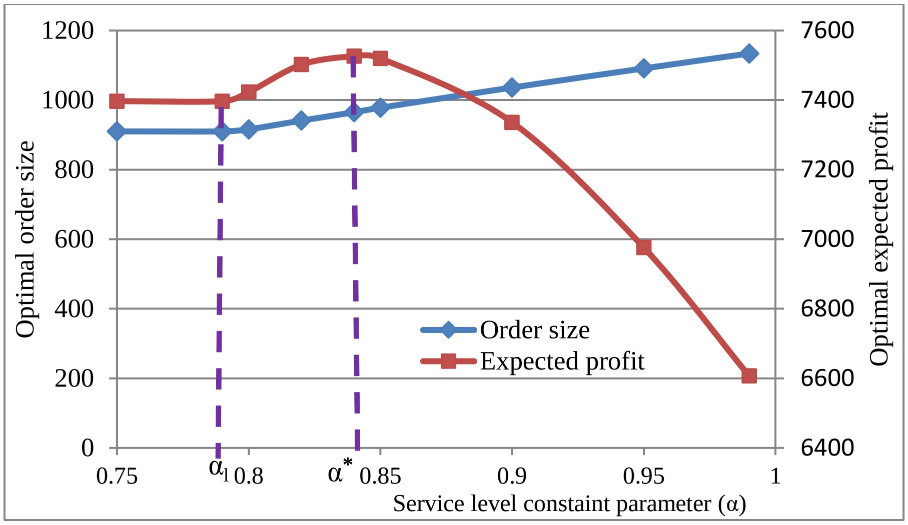

Fig.4. Impact of the service level constraint parameter α on Q_{1c}^{*} and $E(\pi_{1c}^{L})^{*}$

5.5 Problem illustrating General Case

The GC is illustrated using above example data: P = \$35, C = \$20, V = \$12, S = \$5, C_H=15, $\gamma = 1.6$, μ_0 =1000 and σ_0 =200. Three demand adjustment combinations of Δ and δ are presented as three cases along with results (Using Eq.24-26) in Table 4. The case C1 has positive adjustment of mean with reduced SD and provides for the largest profit the largest order size. The C3 case is just opposite that of C1, reduced mean with enlarged SD that gives the order size and the smallest expected profit. The middle C2 case has unchanged mean but enlarged SD provides for moderate results.

Table 4. Data and results for general case of demand adjustments

Case	Δ	Δ_r	δ	δ_r	W_1^*	μ_1	σ_1	Q_1^*	$E(\pi_1^L)^*$
C1	250	0.25	-100	-0.5	0.74	1185	126	1245	16189
C2	0	0	100	0.5	1.00	1000	300	1143	11212
C3	-150	-0.15	50	0.25	0.497	925	225	1032	11036

5.6 Example illustrating Multi-item DF NVP

5.6.1 Basic model

The basic multi-item model is illustrated using three products and relevant economic and demand data is given in Table 5. The budget allocated for buying these three products is \$25000. The results computed with and without constraint are given in the same Table. The total purchase cost and optimal expected profit without constraint are \$30774 and \$22352.The imposition of constraint to comply budget \$25000 has reduced both the optimal order sizes and the expected profits, thereby reducing the total expected profit to \$20497.

Table 5. Data and results for results for multi-item DF NVP

	Economic parameters	Base demand	Without constraint ($\lambda_m = 0$)	With constraint ($\lambda_m^* = 0.53$)

Item (i)	P_i	C_i	V_i	S_i	μ_{0i}	σ_{0i}	Q^*_{0i}	PC_i	π^*_{0i}	Q^*_{1i}	PC_{1i}	π^*_{1i}
P1	37	20	12	5	250	80	292	5844	3189	230	4600	2415
P2	75	30	10	7	100	40	120	3600	3210	101	3030	3069
P3	100	45	20	10	400	150	474	21330	15953	386	17370	15013
Total								30774	22352		25000	20497

5.6.2 Model with revised demand

The example 5.6.1 is extended with consideration of additional data (Δ_i, δ_i, CH_i, and γ_i) to illustrate multi-item model with revised demand. In view of surge in demand seen from the positive demand adjustments of mean, the budget is increased from $25000 to $28000. The additional data and results for CVC, CCVC and GC are presented in Table 5. The CVC provides for the greatest expected profit because of its constant demand variability despite increased mean of demand as compared to CCVC.

Table 5. Data and results for results for multi-item DF NVP with revised demand

	Additional data				CVC (λ^*_m= 0.66)			CCVC (λ^*_m= 0.49)					GC (λ^*_m= 0.56)		
Item (i)	Δ_i	Δ_{ri}	CH_i	γ_i	W^*_{1i}	Q^*_{1i}	π^*_{1i}	W^*_{1i}	Q^*_{1i}	π^*_{1i}	δ_i	δ_{ri}	W^*_{1i}	Q^*_{1i}	π^*_{1i}
P1	60	0.24	4	1.4	0.51	244	3108	0.20	246	3042	-20	-0.25	1	293	4001
P2	30	0.30	12	1.6	1.0	126	3972	0.74	124	3546	15	0.37	0.35	110	3261
P3	100	0.25	20	1.8	0.64	430	17030	0.37	430	16182	30	0.20	0.38	419	16050
Total							24110			22770					23312

6. Case study: Retailing of NX Calendar

A real-life case study is presented to illustrate the approach; it includes the following steps:

- Refine initial demand using expert.
- Identify factors/ events influencing the demand along with their potential impacts.
- Aggregate the impacts and revise the demand forecast.
- Determine the order size based on revised demand using the models developed.

The case pertains to retailing a branded and popular calendar, named as NX by a newspaper vendor, Thane Newspaper Agency (TNA), India. NX calendar (date pad cum almanac) is available in 35 variants (sizes and languages) with annual sales of 20 million units. Its wall hanging regular size calendar (274 x 418mm) in Marathi (regional) language predominates with 80% of the global sales and constitutes the focal item of the case study. The six stalls of TNA not only sell newspaper and magazines, but also make deliveries to the doorsteps of their subscribers (about 4500) in the surrounding area. Occasionally, they sell seasonal items such as calendars leveraging on its customer reach. These stalls act as a *nodal agency* to perform many other functions like delivery of advertisement pamphlets, collection of newspapers, etc.

The NX has a concentrated selling period of one month from mid-December. A large volume of the six stands together enables TNA direct procurement of NX from its manufacturer i.e. Printer and Publisher (PP). The minimum order quantity (MOQ) is 2000 and the additional copies can be obtained in a lot of 100, called as a *bundle*. The early order booking by the first week of August has benefit of lower unit price, however, order modification is allowed as late as 1st November with a penalty. An emergency supplies from a distributor cost it far higher. The selling starts with order receipt in the second half of November and continue until mid- January.

Problem and current method of demand forecasting and ordering

Table 4 gives the demand for the past six years. It may be noted that *demand and sale* are different in case of shortage and an adjustment for lost sales is needed. The lost sale is hardly visible and countable; it is often a subjective estimate. For example, the sale and lost sale estimate for the year 2015 are 3600 and 250 respectively and hence the demand is 3850. Secondly, there is a growing trend in demand at a compound rate of 10.4% from 2140 (2011) to 3500 (2016) as shown in Fig.5. The demand in the year 2011 is rather lower due to free distribution of calendars by hopeful election candidates to Thane Municipal Corporation (TMC) election in February 2012. A large increase in the subsequent year 2012 is attributed to (i) regaining the demand (ii) discontinuation of a practice of selling calendar at nominal price by a leading Marathi newspaper (viz. Maharashtra Times).

Table 4. NX aggregate demand over years

Year	2011	2012	2013	2014	2015	2016
Sale	2140	2600	2920	3200	3600	3440
Q	2500	2600	3200	3200	3600	3700
Excess	360 (16.8%)	-	280 (8.75)	-	-	260 (7.6%)
Shortage (estimated)	-	150 (5.8%)	-	200 (7.8%)	250 (8.33%)	-
Demand	2140	2750	2920	3400	3850	3440

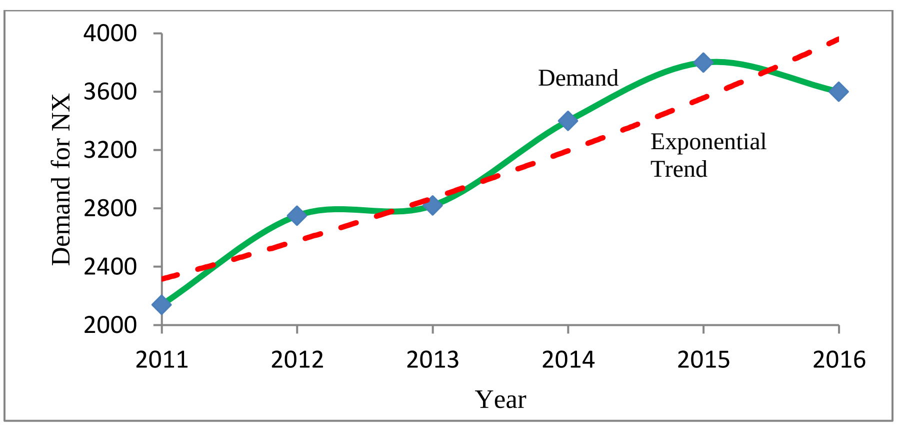

Fig.5.Growing trend in the calendar demand

The forecasting method in the company is mainly based on rule of thumb, experience and intuition. The forecast is determined in consultation with the Stall Managers and considering sales of the previous year only. The forecast proposed by the company for the year 2016 was 3650 units considering the sale of 3600 in the previous year, growth and dip in demand on account for free distribution of calendars in the TMC election year (February 2017). The units ordered were 3700. The difference between the forecast and ordering reaches up to 17% as shown in Table 4. The error may be attributed to the following reasons:

- The current method of forecasting is mainly based on experience and intuition. Further, there is no systematic consideration of some important factors and events.
- The order size is approximated to the demand forecast and is not determined separately.

The Proposed solution

The method outlined in our paper is employed to obtain demand adjustment and order size. The working professional with TNA and manufacturer of NX were consulted in identifying factors/events along with their potential impacts for the year 2016. Table 5 lists the factors and their potential demand impact and provide explanation to some events. A professional from NX pointed out that capacity expansion and subsequent aggressive marketing efforts by its competitor (Janlakshmi Printers) may eventually reduce the demand of NX. Secondly, a few leading Marathi newspapers offer calendar as a compliment at a nominal price. However, two newspapers (Sakal and Deshonnati) have discontinued this practice that would fetch additional demand of 150 to NX. Thirdly, demand is growing as a result of increasing population density and increased penetration. Customers are buying more than one calendar per household/shop for locational convenience and/or buy a specialty calendar, which explains the 7% growth. Fourthly, since there would be local body (TMC) election in February 2017, the free distribution of calendars by hopeful candidates is expected to reduce the demand by 500 units. The sum of potential impacts, i.e. demand adjustment (Δ) is -300. The demand forecast by TNA for the year 2016 is 3650 and accordingly 3700 units ordered. Thus, μ_0=3700, Δ=-300 and Δ_r = - 0.081.

Table 5. Event based factors and their potential impact

Factor category	Event	Potential impact
Quantum jump factor	Discontinuing practice of offering calendar at nominal price by a leading Marathi newspaper	150
	Increase in price of competing brands, Lokmat Kaldarshika, offered by a Marathi newspaper Lokmat to Rs.28	-50
	Capacity expansion and aggressive marketing by competing PP	-100
Trend change factor	Annual growth of 7% is attributed to increasing population density and growing tendency of buying more than one calendar	200
Transient factor	Distribution of calendars by politicians in an election year. This impact correction is needed in the election year of TMC and the year after.	-500
	Demand adjustment (Δ)	-300

Economic parameters

The list price of the calendar is Rs.28 but 25% of the calendars are sold at a discount rate of Rs.25 in initial and trailing selling periods; thus weighted price is Rs.27.25. The unit purchase cost is Rs.15, the penalty for modifying the order size is Rs.3 per unit and therefore $C_H = 3$. The optimal weight factor (W^*) is 1 for given parameters. Selling calendars as waste paper would fetch Rs.2 per unit. The shortage penalty (S) for non-satisfied demand beyond the lost profit is assumed to be zero because of its wide availability. Thus, the model parameters are stated as: P=27.25, C=15, S=0, V=2, C_H= 3, Δ = -300, μ_0=3700, σ_0=350, μ_1=3400, Δ_r = - 0.081, γ =1.5, W^* =1, A (= P-C+S) =12.25, B (=C-V) =13.

Results and discussion

As the distribution underlying the demand is unknown, various cases are considered, however, results for σ_0=350 are presented in Table 6. A case of riskless ordering (σ_0=0) provides upper-bound of expected profit π^*=41650. With W^*=1, σ_0=350 for CVC and σ_1=322 for CCVC. The distribution free CVC has Q^*=3390 and $E(\pi_1^L)^*$ =36333. The results for CCVC: Q^*=3390 and $E(\pi_1^L)^*$ =36691; this profit is more than the CVC because of the lower cost of uncertainty associated with lower standard deviation σ_1=322. Secondly, the order sizes are smaller than the mean because the critical ratio (k) is less than 0.5 (viz. 0.49). Thirdly, the demand following a normal distribution provides for Q^*=3387 and $E(\pi_1)^*$=38339 and following uniform distribution [a=2800, b=4000 from $\sigma_1 = \frac{(a-b)}{\sqrt{12}}$] yield Q^*=3382 and $E(\pi_1)^*$=36965.The normal and uniform demand distributions offer better results than the distribution free case.

Table 6. Order size and expected profit under different demand distribution

	Case	σ_0	σ_1	Q	(P-C) μ_1	Adjustment cost	Cost of uncertainty	Total cost	Profit
1	Riskless	0	0	3400	41650	900	0	900	40750
2	CVC	350	350	3390	41650	900	4417	5317	36333
3	CCVC	350	322	3390	41650	900	4059	4959	36691
4	Normal distribution	350	-	3387	41650	900	3524	4424	37226
5	Uniform distribution	-	-	3382	41650	900	3785	4685	36965

Based on the calculation, we suggested the order size of 3400 (in multiples of 100), which is lower than the previous year demand. The important consideration was the dominant impact of free distribution of calendars in the year of election to local body (TMC), February 2017. However, TNA continued with annual growth optimism and stick to its initial order size of 3700 units. The reported sale in the year 2016 sale was 3440, leaving an excess inventory of 260 units and earned profit Rs. 38240. The TNA would it would have earned a profit of Rs.40425 with order size of 3400, following our advice. Thus, the implementation of proposed method helps in improving the profit by 6.6%.

7. Conclusion

This paper presents models for preseason two-stage ordering. The model addresses potential impact of contextual information collected between two ordering instants through expert judgment and uses to revise the demand forecast. The inventory model under the distribution free NVP setting uses the revised demand to determine the optimal order quantity, the weight factor and the expected profit. The models consider the bidirectional changes in demand change, different cases of variance change and practical limitation. These models are extended to multi-item cases to improve practical utility.

The results show that retailers could expect higher profits through a larger order size in the case of large demand. In addition, retailers benefit from reduced optimal order size in the case of demand contraction, by minimizing the expected cost. Secondly, the enforcement of order size constraint in a demand of enlarged case has reduced both order size as well as expected profit; whereas imposition of service level constraint in a diminished demand case has reduced the expected profit despite raising the order quantity. Thirdly, CCVC embodies higher demand variability than CVC in a demand enlargement case that results in lower expected profit. On the contrary, in a demand contraction case, CCVC embodies lower demand variability than CVC and has more expected profit than the latter. Fourthly, the case study presented shows that the use of the factor-based method to determine the potential impact of contextual information [and the]demand forecast updating and determination of optimal order size is coherent with the proposed models and can lead to profit improvement. However, the case of negative demand adjustment could lead to reduction in expected profit, but it worth considering from the retailer's strategic point of view especially to reduce excess inventory.

The research can be extended to consider a refinement of the factor-based method in obtaining potential impact of contextual information. Secondly, the retailer is considered as a standalone decision maker in isolation. However, it would be more appropriate to consider retailers as a part of a whole supply chain. Therefore, the work can be extended in perspective of supply contract, such as quantity flexibility that allows bidirectional changes of the order quantity. In this way demand risk sharing is achieved and the supply chain between the newsvendor and its supplier can be coordinated.

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Appendix A: Proof of Proposition 1

Common Case (applicable to CVC, CCVC and GC):

The lower bound of the expected objective function is given by

$$E(OF^L) = (\theta P - V)\mu_1 - (C - V)\,Q - (P - V + S)\frac{\sqrt{\sigma_i^2 + (Q_1-\mu_1)^2} - (Q_1-\mu_1)}{2} - C_H\mu_0|\Delta_r|W^\gamma \quad (1A)$$

Substituting $Z_1 = Q_1 - \mu_1$, $A = (P - C + S)$ and $B = (C - V)$ for convenience of presenting and equating the first derivative of (1A) w.r.t. Q to zero yields

$$-B - \frac{(A+B)}{2}\left[\frac{2.Z_1}{2\sqrt{\sigma_i^2 + Z_1^2}} - 1\right] = 0$$

It can be shown that $\frac{\partial^2 E(OF^L)}{\partial^2 Q} < 0$ and proving concavity of $E(OF^L)$ w.r.t. Q. Rearrangement of above equation gives

$$\sqrt{Z_1^{\,2} + \sigma_i^{\,2}} - Z_1 = \frac{\sigma_i B}{\sqrt{AB}} \quad (2A)$$

$$Q_1^* = \mu_1 + \frac{\sigma_i}{2}\left[\frac{A-B}{\sqrt{A\,B}}\right] \quad (3A)$$

Substituting Q_1^* in (1A), we get

$$E(OF^L) = (\theta P - C)\mu_1 - \sigma_1\sqrt{A\,B} - C_H\mu_0|\Delta_r|W^\gamma \quad (4A)$$

For CVC:

The first derivative of (4A) w.r.t. W:

$$\frac{\partial E(OF^L)}{\partial W} = (\theta P - C)\mu_0\Delta_r - C_H\mu_0|\Delta_r|W^{\gamma-1}\gamma$$

It can be shown that $\frac{\partial^2 E(OF^L)}{\partial^2 W} < 0$. Hence $\frac{\partial E(OF^L)}{\partial W}$ =0 would yield optimal weight factor as

$$W_1^* = \left[\frac{(\theta P - C)\Delta_r}{C_H|\Delta_r|.\gamma}\right]^{\frac{1}{\gamma-1}} \quad (5A)$$

The substitution of W_1^* and appropriate θ in (4A) yields optimal $E(OF^L)$.Thus, for $\Delta_r \geq 0$, $(\theta = 1)$ optimal lower bound of expected profit is given by

$$E(\pi_1^L)^* = (P - C)\mu_1 - \sigma_0\sqrt{AB} - C_H\mu_0|\Delta_r|W_1^{*\gamma} \quad (6A)$$

For $\Delta_r < 0$ $(\theta = 0)$ the objective function represents the upper bound of the expected negative cost given by

$$E(CO)^* = -\left(C\mu_1 + \sigma_0\sqrt{AB} + C_H\mu_0|\Delta_r|W_1^{*\gamma}\right) \quad (7A)$$

The addition of expected revenue to 7A) gives an optimal lower bound of expected profit as:

$$E(\pi_1^L)^* = (P-C)\mu_1 - \sigma_0\sqrt{AB} - C_H\mu_0|\Delta_r|W_1^{*\gamma} \quad (8A)$$

For CCVC:

Calculus is used to obtain optimal value of W. In CCVC, $\sigma_i = \sigma_1$ is a function of W and hence $\frac{d\sigma_i}{dW} = \sigma_0\Delta_r$. The second derivative of (4A) w.r.t. W can be shown to be negative and therefore optimal weight factor for CCVC is

$$W_1^* = \left[\frac{(\theta P - C)\mu_0\Delta_r - \sigma_0.\Delta_r.\sqrt{AB}}{C_H|\Delta r|.\gamma.\mu_0}\right]^{\frac{1}{\gamma-1}} \quad (9A)$$

Similar to CVC, lower bound of the expected profit is obtained as

$$E(\pi_1^L)^* = (P-C)\mu_1 - \sigma_1\sqrt{AB} - C_H\mu_0|\Delta_r|W_1^{*\gamma} \quad (10A)$$

For GC:

The Δ_r and δ_r used $\mu_1[=\mu_0(1+W\Delta_r)]$ and $\sigma_1=[\sigma_0(1+W\delta_r)]$ can be different in both sign and magnitude. The W_1^* is obtained by using relation $\frac{dE(OF^L)}{dW}$=0 where $\frac{d\sigma_1}{dW}=\sigma_0\delta_r$ and $\frac{d\mu_1}{dW}=\sigma_0\Delta_r$. The second derivative of (4A) w.r.t. W can be shown to be negative and therefore W_1^* for GC is

$$W_1^*=\left[\frac{(\theta P-C)\mu_0\Delta_r-\sigma_0.\delta_r.\sqrt{AB}}{C_H|\Delta r|.\gamma.\mu_0}\right]^{\frac{1}{\gamma-1}} \tag{11A}$$

Similar to CVC, the lower bound of the expected profit is obtained as

$$E(\pi_1^L)^*=(P-C)\mu_1-\sigma_1\sqrt{AB}-C_H\mu_0|\Delta_r|W_1^{*\gamma}$$

Hence, the result of (3A), (5A), (6A), (9A), (10A) and (11A) completes the proof of Proposition 1.

Appendix B: Proof of Lemmas

Lemma1

Proof. W_1^* from (5A) and (9A) is

$$W_1^*=\begin{cases}\left[\frac{(\theta P-C)\Delta_r}{C_H|\Delta_r|\gamma}\right]^{\frac{1}{\gamma-1}} & For\ CVC\\ \left[\frac{(\theta P-C)\mu_0\Delta_r-\sigma_0\,\Delta_r\sqrt{A\,B}}{C_H\mu_0|\Delta_r|\gamma}\right]^{\frac{1}{\gamma-1}} & For\ CCVC\\ \left[\frac{(\theta P-C)\mu_0\Delta_r-\sigma_0.\delta_r.\sqrt{AB}}{C_H|\Delta_r|.\gamma.\mu_0}\right]^{\frac{1}{\gamma-1}} & For\ GC\end{cases} \tag{1B}$$

The weight factor cannot be more than 1, the net value of terms within the bracket of (1B) must be less than or equal 1 as $\frac{1}{\gamma-1}>1$ viz.

$$\frac{(\theta P-C)\Delta_r}{C_H|\Delta_r|\,\gamma}\le 1,\ \frac{(\theta P-C)\mu_0\Delta_r-\sigma_0.\Delta_r.\sqrt{(P-C+S)(C-V)}}{C_H|\Delta r|.\gamma.\mu_0}\le 1, \text{ and } \frac{(\theta P-C)\mu_0\Delta_r-\sigma_0.\delta_r.\sqrt{AB}}{C_H|\Delta_r|.\gamma.\mu_0}\le 1$$

It provides for the lower limit of C_H

(i) $C_H\ge\frac{(\theta P-C)\Delta_r}{|\Delta_r|\,\gamma}$ for CVC

(ii) $C_H\ge\frac{(P-C-\lambda_1)\mu_0-\sigma_0\sqrt{(A-\lambda_1)(B+\lambda_1)}}{\mu_0\gamma}$ for CCVC

(iii) $C_H\ge\frac{(\theta P-C)\mu_0\Delta_r-\sigma_0.\delta_r.\sqrt{AB}}{C_H|\Delta_r|.\gamma.\mu_0}$ for GC (2B)

The C_H less than specified in (2B) would make W_1^* more than 1 which is unacceptable. Thus, for the given P, C, S, V and γ, lower bound of C_H that would ensure $W_1^*\le 1$ is given in (2B). This completes the proof of Lemma1.

Lemma 2 and 3:

The logic applicable to Lemma 2 and 3 is very similar to the used in Lemma 1 and therefore proof of lemma 2 and 3 is not provided.

Lemma 4

The $h(\lambda_j)$ is a continuous and strictly increasing function in λ_j. The P1 or P2 becomes an unconstrained optimization problem at $\lambda_j=0$ and $h(\lambda_j)<0$ violates either of the constraints. The λ_j must be positive and therefore there exist two distinct values $\lambda_j^1,\lambda_j^2>0$ such that $h(\lambda_j^1)h(\lambda_j^2)<0$.This implies that there exist a unique λ_j such that $h(\lambda_j)=0$.

Appendix C: Proof of Proposition 2

Derivations for a common case:

The Lagrange function (11) for P2 with $Z_{1c}=Q_{1c}-\mu_1$

$$L=(P-V)\mu_1-(C-V)Q_{1c}-(P-V+S)\frac{\sqrt{\sigma_i^2+Z_{1c}{}^2}-Z_{1}c}{2}-C_H\mu_0|\Delta_r|W_{1c}^{\gamma}-\lambda_1[Q_{1c}-(1+\beta)Q_0] \tag{1C}$$

Setting the first derivative of (1C) w.r.t. Q_{1c} to zero provides for constrained optimal order quantity

$-(C - V + \lambda_1) - \frac{(P-V+S)}{2}\left[\frac{Z_{1c}}{\sqrt{\sigma_i^2 + Z_{1c}^2}} - 1\right] = 0$

Let denote $B_1 = (C - V + \lambda_1)$ and $T = (P - V + S) = (A + B)$ and rearranging gives

$$\frac{Z_{1c}}{\sigma_i} = \frac{(T-2\,B_1)}{2\sqrt{B_1(T-B_1)}} \tag{2C}$$

The term $(T - 2\,B_1)$ is equal to $(A - \lambda_1) - (B + \lambda_1)$ and $B_1(T - B_1)$ is equal to $(A - \lambda_1)(B + \lambda_1)$ and hence the optimal order quantity for CVC case is

$$Q_{1c}^* = \mu_1 + \frac{\sigma_i}{2}\left[\frac{(A-\lambda_1) - (B+\lambda_1)}{\sqrt{(A-\lambda_1)(B+\lambda_1)}}\right] \tag{3C}$$

The rearrangement of (2C) gives $\sqrt{\sigma_i^2 + Z_{1c}^2} - Z_{1c} = \frac{\sigma_i B_1}{\sqrt{B_1(T-B_1)}}$. Its substitution along with Q_{1c}^* into (1C) yields

$$L = (P - V)\mu_1 - (C - V + \lambda_1)Q_{1c}^* - \frac{(P-V+S)}{2}\frac{\sigma_i B_1}{\sqrt{B_1(T-B_1)}} - C_H\mu_0|\Delta_r|W_{1c}^{\gamma} + \lambda_1(1+\beta)Q_0 \tag{4C}$$

Different treatment is required to CCVC and CVC to find W_{1c}^* and $E(\pi_{1c}^L)$ as standard deviation is a function of W_{1c} in former, but not in latter.

For CVC:

For CVC, $\mu_1 = \mu_0(1 + W_{1c}\,\Delta_r)$ and $\sigma_1 = \sigma_0$ that provides $\frac{d\mu_1}{dW_{1c}} = \mu_0\Delta_r$ and $\frac{d\sigma_1}{dW_{1c}} = 0$. The differentiation of (4C) w.r.t W_{1c} and setting it to zero gives optimal weight factor as

$$W_{1c}^* = \left[\frac{(P-C-\lambda_1)}{C_H\gamma}\right]^{\frac{1}{\gamma-1}} \tag{5C}$$

A closed form expression for the optimal lower bound of expected profit is obtained by substituting the values of Q_1, $\sqrt{\sigma_i^2 + Z_{1c}^2} - Z_{1c}$ and W_{1c}^* in (1C) and is given below

$$E(\pi_c^L) = (P - C)\mu_1 - \frac{\sigma_i}{2\sqrt{B_1(T-B_1)}}[B(T - 2\,B_1) + TB_1] - C_H\mu_0|\Delta_r|(W_{1c}^*)^{\gamma}$$

The term $B(T - 2\,B_1) + TB_1$ in the above expression is equal to $B(A - \lambda_1) + A(B + \lambda_1)$ and $B_1(T - B_1)$ is equal to $(A - \lambda_1)(B + \lambda_1)$. Therefore, $E(\pi_{1c}^L)^*$ is

$$E(\pi_{1c}^L)^* = (P - C)\mu_1 - \frac{\sigma_0}{2}\left[\frac{B(A-\lambda_1) + A(B+\lambda_1)}{\sqrt{(A-\lambda_1)(B+\lambda_1)}}\right] - C_H\mu_0|\Delta_r|(W_{1c}^*)^{\gamma} \tag{6C}$$

The derivation for CCVC and GC is quite similar to CVC and hence avoided. Thus, the results of (3C), (5C) and (6C) complete the derivations of (12).

Appendix D: Proof of Proposition 3

Common Case:

Lagrange function for P2

$$L = (C - V)Q_{1c} + V\mu_1 + (P - V + S)\frac{\sqrt{\sigma_i^2 + Z_{1c}^2} - Z_{1c}}{2} - C_H\mu_0|\Delta_r|W_{1c}^{\gamma} - \lambda_2[Q - \propto(\mu_1 + \sigma_i Z_n)] \tag{1D}$$

Setting the first derivative of (1D) w.r.t. Q_{1c} to zero provides

$(C - V) + \frac{(P-V+S)}{2}\left[\frac{2Z_{1c}}{2\sqrt{\sigma_i^2 + Z_{1c}^2}} - 1\right] - \lambda_2 = 0$

Let $B_2 = (C - V - \lambda_2)$ and substituting $(P - V + S)$ = A+B and solving provides

$$Z_{1c} = \frac{\sigma_i}{2}\cdot\frac{(T-2B_2)}{\sqrt{B_2(T-2B_2)}} \tag{2D}$$

Given the $(T - 2B_2) = [(A + \lambda_2) - (B - \lambda_2)]$, $B_2(T - 2B_2) = (A + \lambda_2)(B - \lambda_2)$ and $Z_{1c} = Q_{1c} - \mu_1$, the $Q_{1c}{}^*$ is given as

$$Q_{1c}{}^* = \mu_1 + \frac{\sigma_i}{2}\cdot\left[\frac{(A+\lambda_2)-(B-\lambda_2)}{\sqrt{(A+\lambda_2)(B-\lambda_2)}}\right] \tag{3D}$$

We get $\sqrt{\sigma_i^2 + Z_{1c}^2} - Z_1 = \frac{B_2\sigma_1}{\sqrt{B_2(T-B_2)}}$ from (2D) and its substitution in (1D) provides

$$L = [C + \lambda_2(\propto -1)]\mu_1 + \sigma_i\sqrt{B_2(T - B_2)} + C_H\mu_0|\Delta_r|W_{1c}^{\gamma} + \lambda_2 \propto \sigma_i Z_n \tag{4 D}$$

Different treatment is required for CVC and CCVC because of different standard deviations.

Derivations for CCVC:

Setting first derivative of (1D) w.r.t. W_{1c} to zero and using $\frac{d\sigma_1}{dW_{1c}} = \sigma_0\Delta_r$ provides

$$W_{1c}^* = \left[-\frac{[C+\lambda_2(\propto -1)]\mu_0\Delta_r + \left(\sqrt{(A+\lambda_2)(B-\lambda_2)} + \lambda_2 \propto Z_n\right)\sigma_0\Delta_r}{C_H\mu_0|\Delta_r|.\gamma}\right]^{\frac{1}{\gamma-1}};$$

with $\Delta_r < 0$, W_{1c}^* is given by

$$W_{1c}^* = \left[\frac{[C-\lambda_2(1-\alpha)]\mu_0 + \sigma_0\left(\sqrt{(A+\lambda_2)(B-\lambda_2)} + \lambda_2 \propto Z_n\right)}{C_H\mu_0.\gamma}\right]^{\frac{1}{\gamma-1}} \tag{5D}$$

Substituting the values of Q_{1c}^* and W_{1c}^* in (1D) and writing with abbreviations gives optimal upper bound of expected cost as

$$E(co^U) = C\mu_1 + \frac{\sigma_1}{2\sqrt{B_2(T-B_2)}}[B(T - 2B_2) + TB_2] + C_H\mu_0|\Delta_r|{W_{1c}^*}^{\gamma}$$

As earlier, substituting $[B(T - 2B_2) + TB_2] = B(A + \lambda_2) + A(B - \lambda_2)$ and $B_2(T - B_2)$=$(A + \lambda_2)(B - \lambda_2)$ gives upper bound of the cost

$$E(co^U) = C\mu_1 + \frac{\sigma_1}{2}\left[\frac{B(A+\lambda_2)+A(B-\lambda_2)}{\sqrt{(A+\lambda_2)(B-\lambda_2)}}\right] + C_H\mu_0|\Delta_r|{W_{1c}^*}^{\gamma} \tag{6D}$$

The lower bound of the expected profit is obtained by subtracting $E(co^U)$ from the expected revenue ($P\mu_1$) and is

$$E(\pi_{1c}^L) = (P - C)\mu_1 - \frac{\sigma_1}{2}\left[\frac{B(A+\lambda_2)+A(B-\lambda_2)}{\sqrt{(A+\lambda_2)(B-\lambda_2)}}\right] - C_H\mu_0|\Delta_r|{W_{1c}^*}^{\gamma} \tag{7D}$$

The derivation for CVC is simpler and similar to CCVC with $\frac{d\sigma_1}{dW_{1c}} = 0$. Similar method can be used in deriving expressions in the case of GC with $\frac{d\sigma_1}{dW_{1c}} = \sigma_0\delta_r$. The detail proofs are not provided in these two cases to avoid repetition. Thus, the results of (3D), (5D) and (7D) complete the proof of Proposition 3.

Appendix E: Proof of Proposition 4

Derivations for a common case:

The Lagrange function () for P3 with $Z_{1i} = Q_i - \mu_{1i}$

$$L = (\theta P_i - V_i)\mu_{1i} - B_iQ_{1i} - \frac{(A_i+B_i)}{2}\left(\sqrt{\sigma_{1i}^2 + Z_{1i}^2} - Z_{1i}\right) - C_{Hi}\mu_{0i}|\Delta_{ri}|W_{1i}^{\gamma_i} - \lambda_m\left[\sum_{i=1}^{N} C_iQ_{1i} - G\right] \tag{1E}$$

Setting $\frac{\partial L}{\partial Q_i} = 0$ for all i, provides

$$\frac{\partial L}{\partial Q_i} = -B_i - \left(\frac{A_i+B_i}{2}\right) \times \left[\frac{Z_i\frac{dz_i}{dQ_i}}{2\sqrt{{\sigma_i}^2+{Z_i}^2}} - 1\right] - \lambda_mC_i = 0 \quad \left[\therefore \frac{\partial Z_i}{\partial Q_i} = 1\right]$$

Let $B_{1i} = (B_i + \lambda_mC_i)$ and $T_i = A_i + B_i$, the above expression can be

$$Z_{1i} = \frac{\sigma_{1i}(T_i - 2B_{1i})}{2\sqrt{B_{1i}(T_i - B_{1i})}} \tag{2E}$$

But $(T_i - 2B_{1i})$ = $(A_i - \lambda_mC_i) - (B_i + \lambda_mC_i)$ and $B_{1i}(T_i - B_{1i}) = (B_i + \lambda_mC_i)(A_i - \lambda_mC_i)$,

$$Q_{1i}^* = \mu_{1i} + \sigma_{1i}\left[\frac{(A_i-\lambda_mC_i)-(B_i+\lambda_mC_i)}{\sqrt{(A_i-\lambda_mC_i)(B_i+\lambda_mC_i)}}\right] \tag{3E}$$

The substitution of (2E) into (1E) result into

$$L = (\theta P_i - C_i)\mu_i - \frac{\sigma_i}{2\sqrt{(T_i-B_{i1})B_{i1}}}[(T_i - 2B_{1i})B_{1i} + T_i B_{1i}] - C_{Hi}\mu_{0i}|\Delta_{ri}|W_{1i}^{\gamma_i} - \lambda_m\left[\sum_{i=1}^{N} C_iQ_{1i} - G\right]$$

Three cases of CVC, CCVC and GC are dealt separately because of different standard deviation.

For GC

Setting $\frac{\partial L}{\partial W_{1i}} = 0$ and using $\frac{d\sigma_1}{dW_{1i}} = \sigma_0 \delta_r$ provides for

$$W_{1i}^{*} = \left\{ \frac{[\theta P_i - C_i(1+\lambda_m)]\mu_{0i}\,\Delta_{ri} - \sigma_{0i}\,\delta_{ri}\,\sqrt{(A_i - \lambda_m C_i)(B_i + \lambda_m C_i)}}{C_{Hi}\mu_{0i}|\Delta_{ri}|\,\gamma_i} \right\}^{\frac{1}{\gamma_i - 1}} \tag{4E}$$

For $\Delta_{ri} \geq 0$, $\theta = 1$, the profit function with substitution of $Q_{1i}^{*} = \mu_{1i} + \frac{\sigma_{1i}(T_i - 2\,B_{1i})}{2\sqrt{B_{1i}(T_i - B_{1i})}}$ is

$$E\left(\pi_{1i}^{L}\right)^{*} = (P_i - C_i)\mu_i - \frac{\sigma_i}{2\sqrt{(T_i - B_{i1})B_{i1}}}[(T_i - 2\,B_{1i})B_{1i} + T_i\,B_{1i}] - C_{Hi}\mu_{0i}|\Delta_{ri}|W_{1i}^{\gamma_i}$$

$$= (P_i - C_i)\mu_{1i} - \frac{\sigma_{1i}}{2}\left[\frac{B_i(A_i - \lambda_m C_i) + A_i(B_i + \lambda_m C_i)}{\sqrt{(A_i - \lambda_m C_i)\times(B_i + \lambda_m C_i)}}\right] - C_{Hi}\mu_{0i}|\Delta_{ri}|(W_{1i}^{\gamma_i})^{*} \tag{5E}$$

For $\Delta_{ri} < 0$, $\theta = 0$ and $\delta_{ri} > 0$, the profit function with substitution of $Q_{1i}^{*} = \mu_{1i} + \frac{\sigma_{1i}(T_i - 2\,B_{1i})}{2\sqrt{B_{1i}(T_i - B_{1i})}}$ is

$$E\left(CO_{1i}^{U}\right)^{*} = C_i\mu_i + \frac{\sigma_i}{2\sqrt{(T_i - B_{i1})B_{i1}}}[(T_i - 2\,B_{1i})B_{1i} + T_i\,B_{1i}] + C_{Hi}\mu_{0i}|\Delta_{ri}|(W_{1i}^{\gamma_i})^{*}$$

$$E\left(\pi_{1i}^{L}\right)^{*} = P_i\mu_i - E\left(CO_{1i}^{U}\right)^{*}$$

$$= (P_i - C_i)\mu_{1i} - \frac{\sigma_{1i}}{2}\left[\frac{B_i(A_i - \lambda_m C_i) + A_i(B_i + \lambda_m C_i)}{\sqrt{(A_i - \lambda_m C_i)\times(B_i + \lambda_m C_i)}}\right] - C_{Hi}\mu_{0i}|\Delta_{ri}|(W_{1i}^{\gamma_i})^{*} \tag{6E}$$

A similar approach can be followed in CVC with $\frac{d\sigma_1}{dW_{1i}} = 0$ and in CCVC with $\frac{d\sigma_1}{dW_{1i}} = \sigma_0 \Delta_r$ and expressions for W_{1i}^{*} and $E\left(\pi_{1i}^{L}\right)^{*}$ can be obtained. Thus, the results of (3E) to (6E) complete the proof of Proposition 4.