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Primitive abundant and weird numbers with many prime factors

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Abstract

We give an algorithm to enumerate all primitive abundant numbers (PAN) with a fixed Ω, the number of prime factors counted with their multiplicity. We explicitly find all PAN up to $\Omega = 6$, count all PAN and square-free PAN up to $\Omega = 7$ and count all odd PAN and odd square-free PAN up to $\Omega = 8$. We find primitive weird numbers (PWN) with up to 16 prime factors, the largest of which is a number with 14712 digits. We find hundreds of PWN with exactly one square odd prime factor: as far as we know, only five were known before. We find all PWN with at least one odd prime factor with multiplicity greater than one and $\Omega = 7$ and prove that there are none with $\Omega < 7$. Regarding PWN with a cubic (or higher power) odd prime factor, we prove that there are none with $\Omega \leq 7$. We find several PWN with 2 square odd prime factors, and one with 3 square odd prime factors. These are the first such examples. We finally observe that these results are in favor of the existence of PWN with arbitrarily many prime factors.

1 Introduction

Let n be a positive integer, and let $\sigma(n) = \sum_{d|n} d$ be the sum of its divisors. If $\sigma(n) > 2n$, then n is called *abundant*, whereas if $\sigma(n) < 2n$, then n is called *deficient*. *Perfect numbers* are those n for which $\sigma(n) = 2n$. If n is abundant and can be expressed as a sum of distinct proper divisors, then n is called *semiperfect*, or sometimes also *pseudoperfect*. A *weird number* is a positive integer which is abundant but not semiperfect.

If n is abundant and it is not a multiple of a smaller non-deficient number, then n is called a *primitive abundant number*, PAN in this paper. Similarly, a *primitive weird number*, PWN in this paper, is a weird number which is not a multiple of any smaller weird number.

In two papers dating back to 1913 in the American Journal of Mathematics [6, 7], Leonard Eugene Dickson proves that the sets of PAN having any given number ω of distinct prime factors is finite (for even PAN, one also needs to fix the exponent of 2). He then explicitly finds all odd PAN with $\omega \leq 4$, and all even PAN with $\omega \leq 3$ (see also [9, 12] for errata in Dickson's tables). The number of odd PAN with $\omega = 5$ was found in 2017 by Dičiūnas [5] (see also A303933); in 2018, Liddy [13] announced the number of odd PAN with $\omega = 6$. For $\omega \geq 7$ the problem is still open.

Moreover, Dickson's technique was suitable to prove that there exist only finitely many odd perfect numbers for any fixed ω. Indeed, no odd perfect numbers are known until today, and it has not been proved yet that they don't exist. The same problem has been raised by Benkoski and Erdős in 1974 for PWN [4], and also in this case it is not known whether odd PWN exist.

Recently, the search for PAN has become the subject of ongoing research by several independent research groups [5, 8, 13], in particular as a possible approach to settle the question of the existence of odd PWN and odd perfect numbers. Heuristic arguments suggest that PWN with several distinct prime factors must have a particularly small abundance (see also Theorem 4.2), and PAN with several distinct prime factors have very rarely such a small abundance. Since an odd PWN must have necessarily several distinct prime factors, these arguments are in favour of the nonexistence of odd PWN, as well as in favour of the nonexistence of odd perfect numbers, which theoretically must have at least nine distinct prime factors [16] and zero abundance.

Motivated by the above discussion, in this paper we focus on the set of PAN with a given number Ω of prime factors *counted with their multiplicity*. In this way we have been able to explicitly find all PAN with Ω up to 6, to count PAN with $\Omega \leq 7$, and to count odd PAN with $\Omega \leq 8$. These results are resumed on Table 2 and appear to be new (see also OEIS sequences A298157 and A287728).

Because of the above arguments, the novel techniques we develop here for implementing our algorithms could also be useful not only to efficiently investigate the existence of odd PWN but also for a new insight on the question of odd perfect numbers.

Weird numbers were defined in 1972 by Stan Benkoski [3], and appear to be rare: for instance, up to 10^4 we have only 7 of them [18]. Despite this apparent rarity, which is the reason for the name, Benkoski and Erdős [4] proved that the set of weird numbers has positive asymptotic density. A trivial property of weird numbers is the following: if n is weird and p is a prime larger than $\sigma(n)$, then np is weird (see for example [10, page 332]). This property is the reason of the increasing interest on PWN. Benkoski and Erdős were unable to prove that infinitely many PWN exist. In 2015, the third author proved the infiniteness of PWN assuming a weak form of the well known Cramér conjecture [14].

Since only finitely many PWN are known, the list of the first PWN is regularly updated and at the time of writing (November 2018) the first 1161 PWN are known (see [18, A002975]).

Looking for the largest possible PWN is also very interesting. One approach is to consider patterns in the prime factorization of PWN, see [1]. At the time of writing only a few PWN with 6 and 7 distinct prime factors are known [18] .

In this paper we sensibly improve these results. We find hundreds of PWN with more than 6 distinct prime factors. In particular, we find PWN with up to 16 distinct prime factors (Tables 3 and 4). The largest PWN we have found has 16 distinct prime factors and 14712 digits. As far as we know, this is the largest PWN known, the previous one being 5328 digits long [14].

Another strange behavior in the prime decomposition of PWN is the fact that only five PWN with non square-free odd prime factors were known (see OEIS sequence A273815), and no PWN with an odd prime factor of multiplicity strictly greater than two is known.

On the one hand, we explain this fact with Theorem 4.7: there is no PWN m with a quadratic or higher power odd prime factor and $\Omega(m) < 7$; there is no PWN m with two quadratic odd prime factors and $\Omega(m) = 7$; there is no PWN m with a cubic or higher power odd prime factor and $\Omega(m) = 7$. On the other hand, we find hundreds of new PWN with a square odd prime factor (see Table 5 for a selection of them). We find several new PWN with two square odd prime factors, and one with three square odd prime factors (see Table 6). These are the first examples of this kind.

In the following, we describe the methods used in the paper.

In Section 2 we start with a careful analysis of the set A_Ω of PAN m with a fixed number $\Omega = \Omega(m)$ of prime factors counted with their multiplicity. As a corollary of Dickson's theorems [6, 7], these sets are finite (see also Theorem 3.2). The main result in this section is Theorem 2.6: PAN are of the form mp^e, where p is a prime larger than the largest prime factor of m, and m is a deficient number satisfying certain conditions involving the *center* $c(m) = \sigma(m)/(2m-\sigma(m))$ of m (see Definition 2.4). Note that some results in this section are either easy consequences of the definitions or well-known: however, since we use them extensively in Sections 3 and 4, we leave them in the paper.

In Section 3 we face the problem of explicitly computing A_Ω, or some statistics on it, for specific values of Ω. Here we distinguish the square-free case from the general case, since the former appears to be notably simpler than the latter.

Every square-free PAN is then given by $p_1 \cdots p_{k-1} p_k$ for certain primes $p_1 < \cdots < p_k$, and $p_1 \cdots p_i$ is recursively built from $p_1 \cdots p_{i-1}$ by imposing $p_1 \cdots p_i$ deficient. This gives an explicit construction for A_Ω in the square-free case (see Algorithm 1). However, since the condition for $p_1 \cdots p_i$ to be deficient is open (see Proposition 2.3), we need a termination condition. This is done by exploiting Theorem 3.3, stating that if $m = p_1^{e_1} \cdots p_r^{e_r}$ is deficient, then mpq is a PAN for suitable primes p, q. Applying this machinery we explicitly find all square-free PAN with $\Omega \le 6$, count the square-free PAN with $\Omega \le 7$ and count odd square-free PAN with $\Omega \le 8$ (see Table 1 and OEIS sequences A295369 and A287590).

Adapting the techniques to the non square-free case essentially means allowing consecutive primes in $m = p_1, \ldots, p_k$ to be equal, and being more careful in identifying which sequences of primes give origin to PAN. As already said, we explicitly find all PAN with $\Omega \le 6$, count PAN with $\Omega \le 7$, and count odd PAN with $\Omega \le 8$ (see Table 2 and OEIS sequences A298157 and A287728).

In Section 4 we turn to primitive weird numbers. When we search for PWN with k not necessarily distinct prime factors for large k, it is not computationally feasible to find all PAN and then check for weirdness. Therefore, to compute the deficient seed $m = p_1 \cdots p_{k-1}$, we choose an *amplitude* a and restrict p_i to the first a primes larger than $c(p_1 \cdots p_{i-1})$, and p_k to the a largest primes smaller than $c(p_1 \cdots p_{k-1})$. In order to be able to deal with the huge numbers involved, we represent them in a form we call *index sequence* (see Definition 4.4). Finally, in Remarks 4.5, 4.6, and in Section 5, we explain several of the observations we made during our computations, for possibly useful future reference. The new findings in this section are: a large number of PWN with more than 6 and up to 16 distinct prime factors (see Tables 3 and 4), PWN with one and more odd prime factors squared (Table 5 and Table 6), and Theorem 4.7 on patterns for PWN.

The problem of finding a PWN with a cubic or higher power odd prime factor remains still open. This, and other open questions, are listed in Section 5.

All the software we have developed and results of our experiments are available on-line at the GitHub repository `https://github.com/amato-gianluca/weirds`.

2 Deficient, perfect and abundant numbers

In line with [11], we will refer to $\Delta(n) := \sigma(n) - 2n$ as the *abundance* of n, and to $\delta(n) := 2n - \sigma(n) = -\Delta(n)$ as the *deficiency* of n. It is sometimes convenient to use the notation

$\sigma_\ell(n) := \sum_{d|n} d^\ell$ for the sum of the ℓ-th powers of divisors, so that $\sigma_0(n)$ is the number of divisors of n including 1 and n itself, $\sigma(n) := \sigma_1(n)$ represents the sum of the divisors of n, and $\sigma_{-1}(n) = \sigma(n)/n$ is the *abundancy* of n. One can characterize deficient, perfect and abundant numbers respectively by $\sigma_{-1}(n) < 2$, $\sigma_{-1}(n) = 2$, $\sigma_{-1}(n) > 2$.

If $n = p_1^{e_1} \cdots p_k^{e_k}$ with $p_1 < \cdots < p_k$ primes, then for each p_i we can choose an exponent from 0 to e_i to build a divisor of n.

It is well-known that the function σ_ℓ is multiplicative, i.e., $\sigma_\ell(m)\sigma_\ell(n) = \sigma_\ell(mn)$ for arbitrary positive integers m, n with $(m, n) = 1$. Moreover, since $\sigma_\ell(p^e) \le \sigma_\ell(p)^e$, then σ_ℓ is also sub-multiplicative, i.e., $\sigma_\ell(mn) \le \sigma_\ell(m)\sigma_\ell(n)$, for arbitrary positive integers m and n.

If a positive integer is non-deficient (i.e., either perfect or abundant) and all of its proper divisors are deficient, then it is called *primitive non-deficient.* A *primitive abundant number* PAN is a primitive non-deficient number which is also abundant[1].

The following propositions can be easily proved from the definitions, see for instance [6].

Proposition 2.1.

1. *If m is non-deficient and $n \in \mathbb{N}$, $n > 1$, then mn is abundant.*
2. *All perfect numbers are primitive non-deficient.*
3. *If m is abundant and m/p is deficient for all primes $p \mid m$, then m is primitive abundant.*

Proposition 2.2. *Let $m = p_1^{e_1} \cdots p_k^{e_k}$ with $p_1 < \cdots < p_k$. Choose a position $i \le k$ and a prime p such that $(m, p) = 1$. Let $\widetilde{m}$ be the result of substituting $p_i^{e_i}$ with p^{e_i} in the decomposition of m, i.e., $\widetilde{m} = mp^{e_i}/p_i^{e_i}$. Then*

- *if m is abundant or perfect and $p < p_i$, then $\widetilde{m}$ is abundant;*
- *if m is deficient or perfect and $p > p_i$ then $\widetilde{m}$ is deficient.*

Note that, if $m = p_1^{e_1} \cdots p_k^{e_k}$ is primitive abundant and we replace $p_i^{e_i}$ with p^{e_i} for some $p < p_i$, we are not sure whether the positive integer we obtain is primitive abundant (although we know it is abundant). For example, $3^2 \cdot 5 \cdot 7 \cdot 103$ is primitive abundant, but $2^2 \cdot 5 \cdot 7 \cdot 103$ is not, since $2 \cdot 5 \cdot 7$ is primitive abundant. Another example involving square-free numbers is the following: $2 \cdot 7 \cdot 11 \cdot 13$ is primitive abundant but $2 \cdot 5 \cdot 11 \cdot 13$ is not, since $2 \cdot 5 \cdot 11$ is already abundant.

2.1 Adjoining a new coprime factor p^e to a deficient number

Proposition 2.3 ([6, Formula (10)])**.** *Let m be a deficient positive integer, $e \in \mathbb{N}$ and p is a prime such that $(m, p) = 1$. Then*

- *mp^e is abundant if and only if $\sigma(m)/\delta(m) > p^e/\sigma(p^{e-1})$;*
- *mp^e is perfect if and only if $\sigma(m)/\delta(m) = p^e/\sigma(p^{e-1})$;*
- *mp^e is deficient if and only if $\sigma(m)/\delta(m) < p^e/\sigma(p^{e-1})$.*

In Section 3, we will use Proposition 2.3 to build PAN by adjoining one prime factor at a time to a deficient number, chosen as seed. Since the term $\sigma(m)/\delta(m)$ will have a major role in the following, we introduce a more succinct notation.

[1] Some authors define a PAN to be an abundant number with no abundant proper divisors. The two definitions differ on multiples of perfect numbers. For example, $30 = 2 \cdot 3 \cdot 5$ is primitive abundant according to this alternative definition, but not according to ours, since $2 \cdot 3$ is perfect, hence non-deficient.

Definition 2.4 (Center of a deficient number)**.** Given a deficient number m, we call *center* of m the value $c(m) := \sigma(m)/\delta(m)$.

Let m be deficient and p a prime such that $(m,p) = 1$ and $p < c(m)$. By Proposition 2.3, it turns out that mp is abundant. However, it is not guaranteed to be primitive abundant. Consider for example $m = 16$, with $c(m) = 31$. If we take $p = 7$, we have that $16 \cdot 7$ is abundant but not primitive abundant, since $8 \cdot 7$ is abundant, too. Another example, in which all prime numbers occur with multiplicity one, is $m = 2 \cdot 13 \cdot 31 = 806$. Then $5 < c(m) < 6$. If we take $p = 3$, then mp is abundant but not primitive abundant, since $2 \cdot 3 \cdot 13$ is abundant.

The following proposition contains properties that are either well-known or easily proved by algebraic manipulation, see for instance [6].

Proposition 2.5. *The center enjoys the following properties:*

1. $c(m) = \dfrac{2m}{\delta(m)} - 1 = \dfrac{1}{\dfrac{2}{\sigma_{-1}(m)} - 1}$*, for any deficient* $m \in \mathbb{N}$*;*

2. *if* $n > 1$ *and* mn *is deficient, then* $c(mn) > c(m)$*;*

3. *for any prime* p*,* $c(p^e)$ *is increasing in* e *and* $\lim_{e\to+\infty} c(p^e) = \dfrac{p}{p-2}$*;*

4. *if* m *is deficient and* p, q *are primes coprime with* m*,* $q > p > c(m)$*, then* $c(mq) < c(mp)$*.*

We want to give appropriate conditions ensuring that mp is primitive abundant. We know from Proposition 2.1 and Proposition 2.3, that a necessary condition for mp^e to be primitive abundant is $p^e/\sigma(p^{e-1}) > c(m/q)$ for each prime $q \mid m$. Since our aim is to implement a program to enumerate PAN (see Section 3), we would like to reduce the number of tests we need to perform each time. The following will be useful.

Theorem 2.6 (Structure Theorem for PAN)**.** *Let* m *be a deficient number,* $e \in \mathbb{N}$ *and* p *a prime such that* $(m,p) = 1$*. Then* mp^e *is primitive abundant if and only if all of the following conditions hold:*

1. $p^e/\sigma(p^{e-1}) < c(m)$,

2. $p^e/\sigma(p^{e-1}) > \dfrac{\sigma(m)}{\delta(m) + \dfrac{2m}{\sigma(q^\alpha) - 1}}$ *for each* $q^\alpha \mid\mid m$,

3. *either* $e = 1$ *or* $p^{e-1}/\sigma(p^{e-2}) > c(m)$*.*

Proof. When $e > 1$, by Propositions 2.1 and 2.3 we have that mp^e is primitive abundant if and only if $p^e/\sigma(p^{e-1}) < c(m)$, $p^{e-1}/\sigma(p^{e-2}) > c(m)$ and $p^e/\sigma(p^{e-1}) > c(m/q)$ for each prime $q \mid m$. If $e = 1$, we have a similar result without the second condition. We prove that, if $q^\alpha \mid\mid m$, then $c(m/q) = \dfrac{\sigma(m)}{\delta(m) + \dfrac{2m}{\sigma(q^\alpha) - 1}}$. Let $\beta = \sigma(q^\alpha)/\sigma(q^{\alpha-1}) = \dfrac{1 + \cdots + q^\alpha}{1 + \cdots + q^{\alpha-1}}$. We have:

$$
\begin{aligned}
c(m/q) &= \frac{\sigma(m/q)}{\delta(m/q)} = \frac{\sigma(m)/\beta}{2m/q - \sigma(m)/\beta} = \frac{\sigma(m)}{2m\beta/q - \sigma(m)} = \\
&\frac{\sigma(m)}{2m\dfrac{1 + \cdots + q^\alpha}{q + \cdots + q^\alpha} - \sigma(m)} = \frac{\sigma(m)}{2m\left(1 + \dfrac{1}{q + \cdots + q^\alpha}\right) - \sigma(m)} = \frac{\sigma(m)}{\delta(m) + \dfrac{2m}{\sigma(q^\alpha) - 1}} \qquad (1)
\end{aligned}
$$

This concludes the proof. ∎

Since the expression on the r.h.s. of (1) is increasing on $\sigma(q^\alpha)$, we can just keep track of the largest $\sigma(q^\alpha)$ of all the q^α's entirely dividing m. For computational reasons, the following variant of (1) might be more efficient, since it only involves integer numbers:

$$c(m/q) = \frac{\sigma(m) - \dfrac{\sigma(m)}{\sigma(q^\alpha)}}{\delta(m) + \dfrac{\sigma(m)}{\sigma(q^\alpha)}} \tag{2}$$

The following corollary has been already proved in [1]. We give here a different proof based on Theorem 2.6.

Corollary 2.7. *If m is deficient, p is a prime such that $(m,p) = 1$, $p < c(m)$ and $p \geq \sigma(q^\alpha) - 1$ for each $q^\alpha \mid\mid m$, then mp is primitive abundant.*

Proof. By Theorem 2.6, mp is primitive abundant whenever $p > \dfrac{\sigma(m)}{\delta(m) + \dfrac{2m}{\sigma(q^\alpha) - 1}}$ for each $q^\alpha \mid\mid m$. We have

$$\frac{\sigma(m)}{\delta(m) + \dfrac{2m}{\sigma(q^\alpha) - 1}} = (\sigma(q^\alpha) - 1)\frac{2m - \delta(m)}{\delta(m)(\sigma(q^\alpha) - 1) + 2m} < \sigma(q^\alpha) - 1.$$

□

Remark 2.8. Due to the approximations in the previous proof, it is evident that the condition $p \geq \sigma(q^\alpha) - 1$ is sufficient but not necessary. Consider $m = 8$ and $p = 7$. Although $7 < \sigma(8) - 1$, it turns out that $8 \cdot 7$ is primitive abundant.

The test for primitiveness in the case of square-free abundant numbers is particularly simple, given the following:

Corollary 2.9. *If $p_1 < \cdots < p_k$ are primes such that $m = p_1 \cdots p_k$ is deficient, $p > p_k$ is a prime such that mp is abundant, then mp is primitive abundant.*

Proof. Since $p > p_k$, then $(m,p) = 1$. Moreover, for each p_i, we have $p_i \mid\mid m$, and $p \geq \sigma(p_i) - 1 = p_i$. Then, the first case of Proposition 2.3 gives $p < c(m)$. Now the statement follows from Corollary 2.7. □

2.2 Adjoining an arbitrary prime factor to a deficient number

We now consider the case where we start with a deficient number m and multiply it by a prime factor p not necessarily coprime with m. We want to study under which conditions mp is perfect, (primitive) abundant or deficient.

First of all, consider that Proposition 2.3 does not hold when $(m,p) \neq 1$. For example, for $m = 10 = 2 \cdot 5$ we have $c(m) = 9$ but $2 \cdot 5^2$ is deficient. We may change Proposition 2.3 in the following way:

Proposition 2.10. *If m is deficient and p is a prime such that $p^\alpha \mid\mid m$, then*

- *$p\sigma(p^\alpha) < c(m)$ if and only if mp is abundant;*
- *$p\sigma(p^\alpha) = c(m)$ if and only if mp is perfect;*

- $p\sigma(p^\alpha) > c(m)$ *if and only if* mp *is deficient.*

Proof. We have that $\delta(mp) = 2mp - \sigma(mp) = 2mp - \sigma(m)\frac{p^{\alpha+2}-1}{p^{\alpha+1}-1} = 2mp - \sigma(m)(p + \frac{p-1}{p^{\alpha+1}-1}) = \delta(m)p - \sigma(m)\frac{p-1}{p^{\alpha+1}-1} = \delta(m)p - \sigma(m)/\sigma(p^\alpha)$. □

Example 2.11. If $m = 2 \cdot 5$, we have $c(m) = 9$ but $5 \cdot \sigma(5) = 30$ hence $m \cdot 5$ is deficient. If $m = 2 \cdot 5 \cdot 13 \cdot 61 \cdot 67$ we have $5651 < c(m) < 5652$. Since $61 \cdot \sigma(61) = 3782 < c(m)$, we have that $m \cdot 61$ is abundant.

We may also adapt Theorem 2.6 to the case when p is not coprime with m as follows:

Theorem 2.12. *If* m *is deficient and* p *is a prime such that* $p^\alpha \mid\mid m$*, we have that* mp *is primitive abundant if and only if* $p\sigma(p^\alpha) < c(m)$ *and* $p\sigma(p^\alpha) > \dfrac{\sigma(m)}{\delta(m) + \dfrac{2m}{\sigma(q^\beta) - 1}}$ *for each* $q^\beta \mid\mid m$ *with* $q \neq p$.

Proof. By Propositions 2.1 and 2.10, it is immediate that mp is primitive abundant if and only if $p\sigma(p^\alpha) < c(m)$ and $p\sigma(p^\alpha) > c(m/q)$ for each prime $q \mid m$ with $q \neq p$. In the proof of Theorem 2.6 we have shown that $c(m/q) = \dfrac{\sigma(m)}{\delta(m) + \dfrac{2m}{\sigma(q^\beta) - 1}}$. □

It turns out that, just as for Theorem 2.6, it is enough to check the conditions of Theorem 2.12 only for the largest $\sigma(q^\beta)$ among all $q^\beta \mid\mid m$.

3 Enumerating primitive abundant numbers

Theorems 2.6 and 2.12 allow us to devise an algorithm for enumerating PAN or, more generally, primitive non-deficient numbers. We will enumerate PAN on the basis of their factorization. For this reason, when $m = p_1^{e_1} \cdots p_k^{e_k}$, we will always assume $p_1 < \cdots < p_k$. Moreover, we will denote with $\omega(m) := k$ the number of distinct prime factors in m and with $\Omega(m) := e_1 + \cdots + e_k$ the number of prime factors in m counted with their multiplicity.

Note that, if we fix the number of prime factors counted with multiplicity, then enumeration terminates, thanks to the following results.

Lemma 3.1. *Given a positive integer* m *and* $k \geq 0$*, there are only finitely many PAN of the form* mn *with* $(m, n) = 1$ *and* $\Omega(n) = k$.

Proof. We proceed by induction on k. For $k = 0$ the result is trivial, either m is primitive abundant and $n = 1$ or it is not. If $k \geq 1$, we distinguish whether m is deficient or not. If m is not deficient, then mn is never primitive abundant and the lemma holds. If m is deficient, consider an n such that mn is primitive abundant and $\Omega(n) = k$. Then n has the form $p_1^{e_1} \cdots p_\ell^{e_\ell}$ with $p_1 < \cdots < p_\ell$ and $\sum_{i=1}^{\ell} e_i = k$. Since mn is abundant, $\sigma_{-1}(n) > 2/\sigma_{-1}(m)$. However, the abundancy of n is bounded by

$$\sigma_{-1}(n) = \sigma_{-1}(p_1^{e_1}) \cdots \sigma_{-1}(p_\ell^{e_\ell}) \leq \sigma_{-1}(p_1^{e_1}) \cdots \sigma_{-1}(p_1^{e_\ell}) \leq (1/p_1 + 1)^k$$

Therefore, $(1/p_1 + 1)^k > 2/\sigma_{-1}(m)$, i.e., $1/p_1 > \sqrt[k]{2/\sigma(m)} - 1$. Since m is deficient, $\sigma_{-1}(m) < 2$. Hence, the right hand side of this inequality is positive and p_1 is bounded from the above. Given one of the finitely many p_1 satisfying this condition and $e \in \{1, \ldots, k\}$, by inductive hypothesis there are only finitely many n' coprime with mp^e, with $\Omega(n') = k - e$ and such that $mp^e n$ is abundant. Varying p, these cover all possible values of n in the statement of this lemma. □

Theorem 3.2. *For any $k > 1$, there are only finitely many PAN n with $\Omega(n) = k$.*

Proof. This follows immediately from the previous Lemma for $m = 1$. □

We remark that Theorem 3.2 is a corollary of [6, 7] about finiteness of PAN with a fixed number of odd prime factors (counted without multiplicity) and a fixed power of 2.

3.1 Square-free PAN

We consider the special case of enumerating square-free PAN (SFPAN in the rest of the paper) with k prime factors. The more general case of primitive square-free non-deficient numbers is not interesting, since it is well-known that there is only one square-free perfect number which is 6 [11, p. 71].

The algorithm is a recursive procedure which takes a deficient number $m = \bar{p}_1 \cdots \bar{p}_r$ with $\bar{p}_1 < \cdots < \bar{p}_r$ and $r < k$ as input. Initially $m = 1$. If $r < k - 1$, for each prime $p > c(m)$ we consider $\widetilde{m} = mp$, which is deficient by Proposition 2.3, and recursively call the procedure. If $r = k - 1$, then we consider all primes p contained in the possibly empty open interval $(\bar{p}_r, c(m))$. By Corollary 2.9, each number of the form mp is a PAN.

The algorithm needs a stopping condition in the case $r < k-1$, since we cannot actually test all the countably infinite primes $p > c(m)$. We decide to try primes in increasing order, stopping as soon as we find a p such that there are no PAN starting with mp. The complete description may be found in Algorithm 1. The algorithm is easily checked to be correct. Completeness, i.e., the fact that the algorithm finds all SFPAN of the chosen form, will be discussed later.

Algorithm 1: Enumerating SFPAN with k prime factors.

```
   Function sfpan(k: nat, m: nat) is
       Input: k is a natural number; m = p̄_1 ··· p̄_r is a square-free deficient number, with
              p̄_1 < ··· < p̄_r
       Output: all primitive abundant numbers of the form m · p_1 ··· p_k, with
              p̄_r < p_1 < ··· < p_k
       Result: the number of square-free primitive abundant numbers of the form above
1      count ← 0;
2      if k = 1 then
3          foreach p prime s.t. p̄_r < p < c(m) do
4              Print(mp);
5              count ← count + 1
6          end
7          return count
8      else
9          foreach p prime s.t. p > max(p̄_r, c(m)) do
10             innerCount ← sfpan(k-1, mp);
11             if innerCount = 0 then
12                 return count
13             end
14             count ← count + innerCount;
15         end
16     end
   end
```

When we only want to count PAN, steps 3–6 of the algorithm may be replaced by a prime counting function. Using an implementation in SageMath of the algorithm and the prime counting function provided by Kim Walisch's `primecount` library, we managed to count the number of SFPAN from 1 up to 7 distinct prime factors and odd SFPAN from 1 up to 8 distinct prime factors. The result is shown in Table 1 and form sequences A295369 and A287590 of the OEIS.

We have also computed a list of SFPAN with up to 6 distinct prime factors, which is available on GitHub at `https://github.com/amato-gianluca/weirds`.

ω	# all	# odd
1	0	0
2	0	0
3	1	0
4	18	0
5	610	87
6	216054	14172
7	12566 567699	101 053625
8	?	3475 496953 795289

Table 1: Number of SFPAN and odd SFPAN with given number of distinct prime factors.

3.2 Completeness of the enumeration algorithm

The critical point of this algorithm is the stopping condition. Are we sure we do not lose any PAN? In order to ensure completeness of the search procedure, we need to prove that, if there is no SFPAN n such that $\omega(n) = k$ and whose factorization starts with $p_1 \cdots p_r$, then there is no SFPAN m with $\omega(m) = k$ and whose factorization starts with $p_1 \cdots p_{r-1} \cdot p$ for any $p > p_r$. We actually prove the contrapositive, i.e., that if $p_1 \cdots p_{r-1} \cdot p \cdot p_{r+1} \cdots p_k$ is primitive abundant and $p_{r-1} < p_r < p$, then there exists an SFPAN m with $\omega(m) = k$ and whose factorization starts with $p_1 \cdots p_{r-1} \cdot p_r$. Note that $p_1 \cdots p_{r-1} \cdot p_r \cdot p_{r+1} \cdots p_k$ is abundant, but it might not be primitive abundant (see examples after Proposition 2.2).

Since a similar stopping condition will be used also in the algorithm of the next subsection, we will also consider the case of non-necessarily square-free PAN.

Theorem 3.3 (Deficient sequence completion). *If $m = \bar{p}_1^{e_1} \cdots \bar{p}_r^{e_r}$ is deficient and $c(m) \geq \bar{p}_r$, then there are primes p, q with $\bar{p}_r < p < q$ such that mp is deficient and mpq is abundant. If m is square-free, mpq is primitive abundant.*

Proof. First of all, when m is square-free, mpq is primitive abundant by Corollary 2.9.

For the main part of the theorem, we consider initially the case $c(m) \geq 8$. Let p be the smallest prime larger than $c(m)$. Then $p > c(m) \geq \bar{p}_r$, and mp is deficient by Proposition 2.3. We need to find a prime $q > p$ such that mpq is abundant. This requires $q < c(mp)$. We have

$$c(mp) = \frac{\sigma(m)(p+1)}{2mp - \sigma(m)(p+1)} = \frac{\sigma(m)(p+1)}{\delta(m)p - \sigma(m)} = \frac{p+1}{\frac{p}{c(m)} - 1}$$

In 1952, Jitsuro Nagura [15] proved that for any $x \geq 8$ there is always a prime strictly between x and $3x/2$. Therefore, by definition of p, using $x = c(m)$ in Nagura's Theorem, we have $p < 3c(m)/2$ and

$$c(mp) > 2(p+1) = 2p + 2$$

Again by Nagura's Theorem (or even weaker results), there is a prime q in the interval $(p, 2p+2)$. Thus, $q < 2p + 2 < c(mp)$, and this concludes the case $c(m) \geq 8$.

We now consider the case $c(m) < 8$, which implies $\bar{p}_r < 8$.

If $\bar{p}_r = 7$, then $c(m) \geq 7$. Let us take $p = 11$ and $q = 13$. Then mp is deficient and $c(mp) = 12/(11/c(m) - 1) \geq 12/(11/7 - 1) = 21$, hence mpq is abundant.

If $\bar{p}_r = 5$, then $r \neq 1$, because $c(5^{e_r}) < 5/(5-2) = 5/3$ for any e_r by (3) in Proposition 2.5. If both 2 and 3 are other factors of m, then m is abundant by Proposition 2.1, because $2 \cdot 3 \cdot 5$ is abundant. Therefore, m is either of the form $2^{e_1} 5^{e_2}$ or $3^{e_1} 5^{e_2}$ for $e_1, e_2 \geq 1$. Since $c(2 \cdot 5) = 9 > 8$, we only consider the case $m = 3^{e_1} 5^{e_2}$, by (2) in Proposition 2.5. Since $c(3^2 \cdot 5^2) > 8$, the only cases remaining are: $m = 3 \cdot 5$, $m = 3^2 \cdot 5$ and $m = 3 \cdot 5^2$. However, $c(3 \cdot 5) = 4 < 5$ and $c(3 \cdot 5^2) = 62/13 < 5$. Therefore, the only m satisfying the hypothesis of the theorem is $3^2 \cdot 5$, for which we may take $p = 7$ and $q = 11$.

If $\bar{p}_r = 3$, then $r = 1$: if 2 also appears as a prime factor in m, then m cannot be deficient since $2 \cdot 3$ is perfect, see Proposition 2.1. Then $m = 3^{e_1}$ for some e_1. However, $c(3^{e_1}) < 3/(3-2) = 3$, hence m does not satisfy the hypothesis of the theorem.

If $\bar{p}_r = 2$, then $m = 2^{e_r}$ and $\sigma(m) = c(m) = 2^{e_r+1} - 1$. If $e_r \geq 3$ then $c(m) \geq 8$ and we fall into the previous case. For the remaining cases: if $m = 2$, take $p = 5$ and $q = 7$; if $m = 4$, take $p = 11$ and $q = 13$. □

Remark 3.4. In the hypothesis of the previous theorem, when m is not square-free, it might not be possible to obtain p, q such that mpq is primitive abundant. Consider $m = 3^8 \cdot 5$, so that $8 < c(m) < 9$. If we determine p as the smallest prime $p > c(m)$ and q as the largest $q < c(mp)$ as in Proposition 2.3, we get $p = 11, q = 53$ and $m \cdot 11 \cdot 53$ which is abundant but not primitive abundant, since $3^7 \cdot 5 \cdot 11 \cdot 53$ is abundant, too. If we replace 53 with smaller primes q the abundance increases, because in general $\Delta(mq) - \Delta(mq') = \Delta(m)(q - q')$ whenever q, q' are coprime with m, hence $m \cdot 11 \cdot q$ is abundant and the integer we obtain cannot be primitive abundant by Proposition 2.2. By increasing p and computing the corresponding largest possible $q < c(mp)$, we get $m \cdot 13 \cdot 31$ and $m \cdot 17 \cdot 19$, but none of them is primitive abundant. We have $[c(m \cdot 19)] = 17$, hence for primes $p \geq 19$ we get $c(m \cdot p) \geq c(m \cdot 19) > 17$ by (4) in Proposition 2.5, and we have no primes $q > p$ making mpq abundant.

Even relaxing the condition $\bar{p}_r < p < q$ into $\bar{p}_r \leq p \leq q$, we do not get any PAN of the form mpq. Actually, $m \cdot 11^2$ is deficient, hence no integer of the form mpq is abundant when $p = q \geq 11$, by Proposition 2.2. If we take $p = 5$, we have $13 < c(m \cdot 5) < 14$. Hence, $m \cdot 5 \cdot 13$ is abundant, but not primitive abundant, since $3^7 \cdot 5^2 \cdot 13$ is abundant, too. Finally, $m \cdot 5^2$ is not abundant.

Corollary 3.5. *If $m = \bar{p}_1^{e_1} \cdots \bar{p}_r^{e_r}$ is deficient and there exists a prime $p > \bar{p}_r$ such that mp is abundant, then for each $s > 0$ there are primes $p_1 < \cdots < p_s$ such that $p_1 > \bar{p}_r$, $mp_1 \cdots p_s$ is abundant and $mp_1 \cdots p_i$ is deficient for each $i < s$. Moreover, if m is square-free, then $mp_1 \cdots p_s$ is primitive abundant.*

Proof. For $s = 1$ the result follows by choosing $p_1 = p$. For $s > 1$, it follows by repeatedly applying Theorem 3.3. Note that since mp is abundant and $p > \bar{p}_r$, then $c(m) > p > \bar{p}_r$ by Proposition 2.3, hence the hypothesis of Theorem 3.3 hold and they are preserved by repeated applications. The result for m square-free follows from Corollary 2.9. □

If m is not square-free, the fact that $p_1, \ldots, p_s$ may be chosen in such a way that $p_1, \ldots p_s$ is primitive abundant is not always true: take for instance $m = 3^8 \cdot 5$ as in Remark 3.4. Then $m \cdot 7$ is abundant, but we have seen there are no p, q such that $5 \leq p \leq q$ and mpq is primitive abundant.

The following theorem proves that the algorithm enumerating SFPAN is complete.

Theorem 3.6. *Let $m = p_1 \cdots p_k$ be an abundant number, with $p_1 < \cdots < p_k$. Let $j < k$ and $p_{j-1} < \tilde{p}_j < p_j$ such that $p_1 \cdots p_{j-1} \tilde{p}_j$ is deficient. Then, there are primes $\tilde{p}_{j+1} < \cdots < \tilde{p}_k$ such* *that* $\tilde{p}_j < \tilde{p}_{j+1}$*,* $\widetilde{m} = p_1 \cdots p_{j-1}\tilde{p}_j \cdots \tilde{p}_k$ *is primitive abundant and* $p_1 \cdots p_{j-1}\tilde{p}_j \cdots \tilde{p}_i$ *is deficient for every* $i < k$*.*

Proof. Let r be the first index such that $p_1 \cdots p_{j-1}\tilde{p}_j p_{j+1} \cdots p_r$ is abundant. Then $r > j$ by hypothesis, and $r \leq k$ because $m\tilde{p}_j/p_j$ is abundant by Proposition 2.2. Then we just apply Corollary 3.5 in order to adjoin $k - r + 1$ prime factors to $p_1 \cdots p_{j-1}\tilde{p}_j p_{j+1} \cdots p_{r-1}$. □

3.3 Possibly non square-free PAN

An extension of the algorithm to find (non necessarily square-free) PAN n with a fixed $\Omega(n)$ may be devised by allowing consecutive prime factors to be equal.

In other words, we see a positive integer m as the product of primes $\bar{p}_1, \ldots, \bar{p}_r$ with $\bar{p}_1 \leq \cdots \leq \bar{p}_r$. When called with $r < k - 1$, the recursive procedure tries to extend m to a deficient number $\widetilde{m} = mp$ using either $p = \bar{p}_r$ or $p > c(m)$ as for the square-free case. When $r = k - 1$, the procedure tries to obtain an abundant number mp by choosing either $p = \bar{p}_r$ or $p < c(m)$. In both cases, when $p = \bar{p}_r$, Proposition 2.10 is used to decide whether mp is abundant or deficient.

In the square-free case, when $r = k - 1$, it is enough to choose $p > \bar{p}_r$ in order to ensure that mp is not only abundant, but also primitive abundant. In the non square-free case this is not enough: we need to use a different lower bound for p, which can be computed using Theorem 2.12.

Another difference with respect to the square-free case is the stopping condition. The reason lies in the extension of Theorem 3.6 to possibly non square-free number.

Theorem 3.7. *Let* $m = p_1 \cdots p_k$ *be a PAN, with* $p_1 \leq \cdots \leq p_k$*. Let* $j < k$ *and* $p_{j-1} < \tilde{p}_j < p_j$ *such that* $p_1 \cdots p_{j-1}\tilde{p}_j$ *is deficient. Then, there are primes* $\tilde{p}_{j+1} \leq \cdots \leq \tilde{p}_k$ *such that* $\tilde{p}_j \leq \tilde{p}_{j+1}$*,* $\widetilde{m} = p_1 \cdots p_{j-1}\tilde{p}_j \cdots \tilde{p}_k$ *is abundant and* $p_1 \cdots p_{j-1}\tilde{p}_j \cdots \tilde{p}_i$ *is deficient for every* $i < k$*.*

Proof. Since σ_{-1} is sub-multiplicative, if we replace in m the prime p_j with $\tilde{p}_j$, the resulting integer $m\tilde{p}_j/p_j$ is abundant. Actually $2 < \sigma_{-1}(m) = \sigma_{-1}(mp_j/p_j) \leq \sigma_{-1}(m/p_j)\sigma_{-1}(p_j) \leq \sigma_{-1}(m/p_j)\sigma_{-1}(\tilde{p}_j) = \sigma_{-1}(m\tilde{p}_j/p_j)$. Let r be the first index (which by hypothesis is strictly larger than j) such that $p_1 \ldots p_{j-1}\tilde{p}_j p_{j+1} \cdots p_r$ is abundant. Then we just apply Corollary 3.5 in order to add $k - r + 1$ prime factors to $p_1 \ldots p_{j-1}\tilde{p}_j p_{j+1} \cdots p_{r-1}$. □

We cannot guarantee that $\widetilde{m}$ is primitive abundant. For example, although $3^6 \cdot 5 \cdot 13 \cdot 31$ is primitive abundant, there is no $p \geq 11$ such that $m = 3^6 \cdot 5 \cdot 11 \cdot p$ is primitive abundant.

Since Theorem 3.7 does not ensure that $\widetilde{m}$ is primitive abundant, the procedure should return a boolean saying whether an abundant number (not necessarily a primitive abundant number) has been found, and stop when the recursive call returns false.

The complete description may be found in Algorithm 2. Using an implementation of the algorithm in SageMath we managed to count the number of PAN with 1 to 7 prime factors (counted with their multiplicity) and odd PAN with 1 to 8 prime factors. The results are shown in Table 2 and form sequences A298157 and A287728 in the OEIS. We have also computed a list of PAN with up to 6 prime factors, which is available on GitHub at `https://github.com/amato-gianluca/weirds`.

4 Weird numbers

In a previous paper [1], we developed search algorithms which allowed us to find primitive weird numbers (PWN) with up to 6 different prime factors. However, we were not able to proceed further due to the computational complexity involved. It was clear that a different approach was needed, which was suggested to us by the following result.

Algorithm 2: Enumerating primitive non-deficient numbers with k prime factors, counted with their multiplicity.

Function `pndn(`*k: nat, m: nat = 1*`)` **is**

Input: k is a natural number; $m = \bar{p}_1^{e_1} \cdots \bar{p}_r^{e_r}$ is a deficient number with $\bar{p}_1 < \cdots < \bar{p}_r$

Output: all primitive non-deficient numbers of the form $m \cdot p_1 \cdots p_k$, with $\bar{p}_r \leq p_1 \leq \cdots \leq p_k$.

Result: a pair (count, found) where count is the number of primitive non-deficient number of the form above, and found is a boolean which is true when a (possibly non-primitive) non-deficient number of the form above has been found.

```
begin
    count ← 0;
    found ← false;
    if k = 1 then
        if there is a prime p s.t. p̄_r < p ≤ c(m) then
            found ← true;
            lowerbound ← max{c(m/p) | p is a divisor of m};
            foreach p prime s.t. max(p̄_r, lowerbound) < p ≤ c(m) do
                Print(mp);
                count ← count + 1
            end
        end
        if p̄_r · σ(p̄_r^{e_r}) ≤ c(m) then
            found ← true;
            lowerbound ← max{c(m/p) | p < p̄_r is a divisor of m};
            if p̄_r · σ(p̄_r^{e_r}) > lowerbound then
                Print (m p̄_r);
                count ← count +1
            end
        end
        return (count,found)
    else
        if m p̄_r is deficient then
            (innerCount, innerFound) ← pndn(k − 1, m p̄_r);
            count ← count + innerCount;
            found ← found or innerFound
        end
        foreach p prime s.t. p > max(p̄_r, c(m)) do
            (innerCount, innerFound) ← pndn(k − 1, mp);
            if innerFound = false then
                return (count, found)
            end
            count ← count + innerCount;
            found ← found or innerFound;
        end
    end
end
end
```

Ω	# all	# odd
1	0	0
2	0	0
3	2	0
4	25	0
5	906	121
6	265602	15772
7	13232731828	102896101
8	?	3475842606319962

Table 2: Number of PAN and odd PAN with given number of prime factors counted with multiplicity.

Proposition 4.1. *A positive integer is primitive weird if and only if it is weird and primitive abundant.*

Proof. If n is primitive weird, by definition it is weird and abundant. We prove that any divisor $m \mid n$, $1 \leq m < n$, is deficient. Assume $n = mk$ with $k > 1$. For the sake of contradiction, assume m is non-deficient. Since m cannot be weird by hypothesis (n primitive weird), there is a subset S of divisors of m such that $m = \sum_{d \in S} d$. If d is a divisor of m, dk is a divisor of n. Hence $n = mk = \sum_{d \in S} dk$ is not weird, contradicting our hypothesis.

On the other hand, let n be weird and primitive abundant. If $m \mid n$ then m is deficient, hence it cannot be weird. Therefore, n is primitive weird. □

Given that PWN are only a particular case of PAN, we use the algorithms for enumerating PAN shown in the previous section, and add a straightforward check for weirdness, transforming them into algorithms for enumerating PWN. Checking for weirdness can actually be made more efficient using the following well-known fact, see [14, Lemma 2].

Proposition 4.2. *An abundant number n is weird if and only if $\Delta(n)$ cannot be expressed as a sum of distinct proper divisors of n.*

4.1 The square-free case

We consider again Algorithm 1 for the square-free case. Since we are interested in finding PWN with several prime factors, and since it is not computationally feasible to enumerate all PAN in such cases, we provide as an additional input to the algorithm an *amplitude* value a. At each step of the procedure, when iterating over primes larger than $c(m)$ (or smaller than $c(m)$ in the case $r = k - 1$), we only consider at most the first a primes.

Another generalization consists in starting the search procedure from a possibly non square-free deficient number m. This means that, in Algorithm 1, each $\bar{p}_i$ could be replaced by a higher power of that prime number, while new factors added by the procedure would remain square-free. However, when $r = k - 1$, we only consider primes p which are larger than $\sigma(q^\alpha)$ for each $q^\alpha \mid\mid m$. In that way, by Corollary 2.7, the abundant numbers found by the search procedure turn out to be primitive abundant. When m is a power of 2, $c(m) = \sigma(m)$ and there are no additional constraints on the choice of the last prime.

Remark 4.3. In determining whether a number is weird, the sufficient conditions in Theorem 3.1 of our previous paper [1] could be employed. However, experimental evaluation has shown that most of the weird numbers generated with our approach fail to satisfy these conditions. Therefore, a direct proof of weirdness using Proposition 4.2 is employed.

The weird numbers generated by this procedure tend to be huge. At each step, since we choose p close to $c(m)$, we minimize the deficiency of $\widetilde{m} = mp$. However, when recursively calling the search procedure on $\widetilde{m}$, since $\delta(\widetilde{m})$ is small, $c(\widetilde{m})$ is quite large. This is repeated step after step, leading to very large prime factors. For example, all the PWN we have generated with $\omega = 12$ are larger than 10^{900}. Since dealing with these huge numbers is cumbersome, we represent them in a form we have called *index sequence*, that turned out to be very useful.

Definition 4.4 (Index sequence)**.** Given a positive integer $m = p_1^{e_1} \cdots p_k^{e_k}$, $p_1 < \cdots < p_k$, such that none of the partial products $w_i := \prod_{j=1}^{i} p_j^{e_j}$ is perfect, we define $\iota(m)$, the *index sequence* associated to m, as the sequence $[(\iota_1, e_1), \ldots, (\iota_k, e_k)]$ with $\iota_1, \ldots, \iota_k \in \mathbb{Z}$ such that:

- if $\iota_i = 0$, then $p_i = c(w_{i-1})$;
- if $\iota_i > 0$, then p_i is the ι_i-th prime larger than $c(w_{i-1})$;
- if $\iota_i < 0$, then p_i is the $|\iota_i|$-th prime smaller than $c(w_{i-1})$, *i.e.*, the result of applying $|\iota_i|$ times the "previous prime" function $\mathrm{pp}(x) := \max\{p < x; p \text{ prime}\}$.

To ease notation, we write a pair (ι, e) as ι^e, or just ι if $e = 1$.

For example, the number $m = 2^2 \cdot 13 \cdot 17 \cdot 443 \cdot 97919 \cdot 563915507$ is represented by the index sequence $[1^2, 2, 1, 1, 1, -2]$, because 2 is the first prime larger than $c(1) = 1$, 13 is the second prime larger than $c(2^2) = 7$, 17 is the first prime larger than $c(2^2 \cdot 13) = 16.\bar{3}$, and so on. All index sequences generated by our search procedure have positive indices for all but the last position. All the indices have an absolute value smaller than the amplitude parameter a.

Remark 4.5. Having to deal with huge integers is a limitation of our approach: increasing the value of k has a big impact on performance because not only is the search space increased by a factor a (the amplitude of the search space) but the integers we deal with also become much larger. Experimentally we see that, when p_i is near $c(p_1^{e_1} \cdots p_{i-1}^{e_{i-1}})$, then each prime is roughly double the size of the preceding one, in terms of the number of digits. Therefore, there is an exponential increase in the size of factors, which impacts all operations on these numbers, but particularly the procedure for determining the (pseudo-)prime immediately preceding or following a given n. This procedure essentially works by repeatedly calling a (pseudo-)primality test with consecutive odd numbers until a new (pseudo-)prime is found. Since in the average the gap between primes is $\log n$ and the BailliePSW primality test [2, 17] used by SageMath takes time proportional to $\log^3 n$, the computational complexity of determining the next prime is roughly $\log^4 n$, i.e., 4^k. This makes it extremely hard to run our algorithms with values of $\Omega > 16$, even with a small value for the amplitude.

On the other side, it seems that the abundant numbers m generated in this way are very likely going to be weird. This is, at least in part, due to the fact that $\Delta(m)$ is low if compared to m and its prime factors. A low abundance is unlikely to be expressible as sum of divisors of m, see Proposition 4.2.

In line with the previous remark, many PWN are easily found starting from a power of two for m and a small amplitude for a. Tables 3 and 4 contain some of the PWN we have found starting from the following parameters:

- $m = 2$, $a = 8$, $k \in \{3, \ldots, 10\}$;
- $m = 4$, $a = 3$, $k \in \{3, \ldots, 16\}$;
- $m = 8$, $a = 6$, $k \in \{3, \ldots, 10\}$.

Table 3 contains, for each PWN, both its factorization and its index sequence. Table 4 only contains index sequences since the constituent primes would not fit on the page. In particular, we mention the following results:

- We have found PWN with up to 16 distinct prime factors. Previously, PWN with 6 distinct prime factors were shown in [1], and later a few more with 6 and 7 distinct prime factors was provided in [18, A002975], while no PWN was known with 8 or more distinct prime factors.
- The PWN with 16 distinct prime factors has 14712 digits. This is, to the best of our knowledge, the largest PWN known, the previously largest having 5328 digits [14].

Note that, for the sake of efficiency, the search algorithm uses pseudo-primes. However, all the factors for the weird numbers in Tables 3, 4, 5 and 6 have been proved to be true primes, for some of them using additional software such as Primo (a primality proving program based on the Elliptic Curve Primality Proving algorithm).

Remark 4.6. Explaining the fact we find so many PWN only on the basis of their abundance is not satisfactory. In particular, by looking at the tables, it is evident that the initial value $m = 4$ is the best choice for searching PWN, at least for small values of the amplitude parameter: with a value of just $a = 3$, we could find PWN with k distinct prime factors for all k between 3 and 16. The results for $m = 2$ and $m = 8$ were less satisfactory, even using much larger values for the parameter a. We will investigate this behavior in a forthcoming paper.

4.2 PWN with square factors

Another weirdness in the realm of weirds is the rarity of PWN with odd prime factors of multiplicity greater than one. To the best of our knowledge, up to now there were only five known PWN with a square odd prime factor, listed in the OEIS sequence A273815, and no PWN with an odd prime factor of multiplicity strictly greater than two is known.

Using an extension of Algorithm 2 we have found hundreds of new PWN with at least one odd prime factor of multiplicity greater than one. A selection of them may be found in Table 5. We find that there are no such PWN for $\Omega < 7$, and the list for $\Omega = 7$ is complete. From $\Omega = 8$ onwards, our list is only partial. None of the PWN we have found has odd prime factors with exponent greater than two.

On the other side we have found many PWN which have two odd prime factors with exponent greater than one, which were not known up to now. One of them is:

$$2^2 \cdot 13 \cdot 17 \cdot 449 \cdot 24809 \cdot 223797481 \cdot 13437522702621389^2 \cdot \\ 30744384018779243589022128598 97^2 \cdot \\ 1440385376937298918762840234913998065047753753438868 78276167$$

whose index sequence is

$$[1^2, 2, 1, 2, 1, 1, 1^2, 1^2, -1]$$

Other PWN with 2 square odd prime factors are given in Table 6. Actually, the last of them has 3 square odd prime factors, so it is likely that there are weird numbers with any number of square odd prime factors, provided Ω is big enough.

All of the above can be summed up in the following theorem:

Theorem 4.7 (PWN with non square-free odd part and $\Omega \leq 7$)**.** *There is no PWN m with a quadratic or higher power odd prime factor and $\Omega(m) < 7$. There is no PWN m with two quadratic odd prime factors and $\Omega(m) = 7$. There is no PWN m with a cubic or higher power odd prime factor and $\Omega(m) = 7$.*

We recall that the completeness of our search comes from that of Algorithm 2. As explained in Section 3.3, the search method of this algorithm is exhaustive because it recursively extends the seed m with all possible prime factors within the bounds computed according to the respective theorems.

5 Open problems

By examining Tables 5 and 6, together with other weird numbers found by our search procedure and which may be found on-line, we observe some facts which can be useful for further experiments.

First of all, there are some prefixes in the factorization which occur in many PWN. One of this recurring prefix is $2^2 \cdot 13 \cdot 17 \cdot 443 \cdot 97919$, which also leads to many PWN with 2 or more square odd prime factors. PWN with 2 square odd prime factors begin to appear in the results of the search procedure when $\Omega = 12$, and become quite common when $\Omega = 14$. It seems that increasing Ω makes the appearance of this kind of PWN easier. Since our search space is quite restricted, there are probably PWN with 2 square odd prime factors even for $\Omega < 12$, but we think they are quite rare. The same thing may be said about PWN with 3 square odd prime factors, which only appear with $\Omega = 15$. Unfortunately, with $\Omega > 15$ the numbers become huge (thousands of digits) and this makes experiments much more difficult.

Open Question 5.1. Given $n \in \mathbb{N}$, is there a PWN with exactly n square odd prime factors?

If such PWN exist, we define Ω_n as the least possible Ω. From the previous section and Theorem 4.7 we obtain $\Omega_1 = 7$, $8 \leq \Omega_2 \leq 12$, $8 \leq \Omega_3 \leq 15$.

As mentioned, another question is the following.

Open Question 5.2. Is there a PWN with a cubic or higher power odd prime factor?

From the experiments, odd square prime factors seems more common at the right end of the factorization, although in our search results they never appear in the last position.

Open Question 5.3. Is there a PWN which has its largest prime factor squared or to a higher power?

The question is intriguing since we don't know any reason why such PWN should not exist, but none is known so far. Finding one could give a hint on their density, which might also have an impact on the search of odd weird numbers.

In sequence A002975 of the OEIS it was asked in 2014 whether the following fact is true: a weird number is primitive if and only if divided by its largest prime factor it is not weird. The following would be a counterexample:

Open Question 5.4. Is there a weird number w which is not primitive and such that $w/\text{gpf}(w)$ is not weird, where $\text{gpf}(w)$ denotes the largest prime factor of w?

This would settle the above question waiting for an answer since 5 years.

The following problem appears as an editor's comment in [3]. Erdős offered \$25 for its solution.

Open Question 5.5. Is $\sigma(m)/m$ bounded when m ranges through the set of (not necessarily primitive) weird numbers?

Finally, the following would settle a long-standing problem.

Open Question 5.6. Find an odd weird number, or prove that all weird numbers are even.

The above problem was raised by Erdős, who offered \$10 for an example of an odd weird number, and \$25 for a proof that none can exist [3]. Wenjie Fang and Uwe Beckert proved, using parallel tree search, that there are no odd weird numbers up to 10^{21}, and no odd weird numbers up to 10^{28} with abundance not exceeding 10^{14} [8, Section 4.2].

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ω	**factored weird number**	$\mathbf{\Delta(w)}$	**index sequence**
3	$2 \cdot 5 \cdot 7$	4	$[1, 1, -1]$
3	$4 \cdot 11 \cdot 19$	8	$[1^2, 1, -1]$
3	$8 \cdot 17 \cdot 127$	16	$[1^3, 1, -2]$
3	$8 \cdot 19 \cdot 71$	16	$[1^3, 2, -2]$
3	$8 \cdot 19 \cdot 61$	56	$[1^3, 2, -4]$
3	$8 \cdot 23 \cdot 43$	16	$[1^3, 3, -1]$
3	$8 \cdot 29 \cdot 31$	16	$[1^3, 4, -1]$
4	$2 \cdot 5 \cdot 11 \cdot 53$	4	$[1, 1, 1, -1]$
4	$2 \cdot 5 \cdot 13 \cdot 31$	4	$[1, 1, 2, -1]$
4	$4 \cdot 11 \cdot 23 \cdot 251$	8	$[1^2, 1, 1, -1]$
4	$4 \cdot 11 \cdot 23 \cdot 241$	88	$[1^2, 1, 1, -2]$
4	$4 \cdot 11 \cdot 31 \cdot 67$	8	$[1^2, 1, 3, -1]$
4	$4 \cdot 13 \cdot 17 \cdot 439$	8	$[1^2, 2, 1, -1]$
4	$8 \cdot 17 \cdot 137 \cdot 9311$	16	$[1^3, 1, 1, -1]$
4	$8 \cdot 17 \cdot 139 \cdot 4723$	16	$[1^3, 1, 2, -1]$
4	$8 \cdot 19 \cdot 79 \cdot 1499$	16	$[1^3, 2, 1, -1]$
4	$8 \cdot 19 \cdot 83 \cdot 787$	16	$[1^3, 2, 2, -1]$
4	$8 \cdot 23 \cdot 67 \cdot 139$	16	$[1^3, 3, 5, -1]$
5	$2 \cdot 5 \cdot 11 \cdot 59 \cdot 647$	20	$[1, 1, 1, 1, -1]$
5	$4 \cdot 11 \cdot 23 \cdot 257 \cdot 13003$	8	$[1^2, 1, 1, 1, -1]$
5	$4 \cdot 11 \cdot 23 \cdot 257 \cdot 13001$	88	$[1^2, 1, 1, 1, -2]$
5	$4 \cdot 11 \cdot 23 \cdot 263 \cdot 6047$	88	$[1^2, 1, 1, 2, -1]$
5	$4 \cdot 13 \cdot 17 \cdot 449 \cdot 24799$	232	$[1^2, 2, 1, 2, -1]$
5	$4 \cdot 13 \cdot 23 \cdot 61 \cdot 1657$	8	$[1^2, 2, 3, 2, -1]$
5	$8 \cdot 17 \cdot 137 \cdot 9337 \cdot 3953791$	272	$[1^3, 1, 1, 3, -1]$
5	$8 \cdot 17 \cdot 137 \cdot 9341 \cdot 3346951$	16	$[1^3, 1, 1, 4, -1]$
5	$8 \cdot 17 \cdot 137 \cdot 9341 \cdot 3346883$	7088	$[1^3, 1, 1, 4, -6]$
5	$8 \cdot 23 \cdot 47 \cdot 1091 \cdot 107209$	976	$[1^3, 3, 1, 2, -1]$
5	$8 \cdot 23 \cdot 47 \cdot 1103 \cdot 51839$	368	$[1^3, 3, 1, 5, -1]$
5	$8 \cdot 23 \cdot 71 \cdot 127 \cdot 6689$	16	$[1^3, 3, 6, 1, -1]$
5	$8 \cdot 31 \cdot 37 \cdot 163 \cdot 186959$	16	$[1^3, 5, 2, 1, -1]$
5	$8 \cdot 37 \cdot 43 \cdot 67 \cdot 15227$	16	$[1^3, 6, 5, 1, -1]$
6	$4 \cdot 11 \cdot 23 \cdot 269 \cdot 4003 \cdot 24766559$	88	$[1^2, 1, 1, 3, 1, -1]$
6	$4 \cdot 11 \cdot 23 \cdot 269 \cdot 4013 \cdot 1508909$	248	$[1^2, 1, 1, 3, 3, -1]$
6	$4 \cdot 13 \cdot 17 \cdot 443 \cdot 97919 \cdot 563915507$	1768	$[1^2, 2, 1, 1, 1, -2]$
6	$4 \cdot 13 \cdot 17 \cdot 443 \cdot 97931 \cdot 330611657$	4888	$[1^2, 2, 1, 1, 3, -1]$
6	$8 \cdot 17 \cdot 137 \cdot 9349 \cdot 2561627 \cdot 3280965162749$	272	$[1^3, 1, 1, 6, 1, -1]$
6	$8 \cdot 17 \cdot 137 \cdot 9349 \cdot 2561651 \cdot 252384300173$	272	$[1^3, 1, 1, 6, 3, -1]$
6	$8 \cdot 17 \cdot 139 \cdot 4783 \cdot 389749 \cdot 8454956717$	7088	$[1^3, 1, 2, 5, 2, -1]$
6	$8 \cdot 23 \cdot 47 \cdot 1087 \cdot 167863 \cdot 197246914559$	16	$[1^3, 3, 1, 1, 1, -1]$
7	$2 \cdot 5 \cdot 11 \cdot 89 \cdot 167 \cdot 829 \cdot 7972687$	20	$[1, 1, 1, 8, 6, 1, -1]$
7	$4 \cdot 13 \cdot 17 \cdot 443 \cdot 97919 \cdot 563915549 \cdot 10965542434977103$	1768	$[1^2, 2, 1, 1, 1, 2, -1]$

Table 3: Some PWN found by our search algorithm. The first column is the number of prime factors. For each ω, entries are in lexicographic order with respect to the index sequence.

ω	**index sequence**	ω	**index sequence**
7	$[1^3, 1, 2, 2, 1, 4, -1]$	11	$[1^2, 2, 1, 1, 1, 1, 1, 1, 2, 1, -1]$
7	$[1^3, 6, 6, 1, 1, 6, -6]$	11	$[1^2, 2, 1, 1, 1, 1, 1, 3, 1, 3, -1]$
7	$[1^3, 6, 6, 1, 3, 2, -5]$	11	$[1^2, 2, 1, 1, 1, 2, 1, 2, 2, 2, -3]$
7	$[1^3, 6, 6, 1, 3, 2, -6]$	11	$[1^2, 2, 1, 1, 1, 2, 2, 1, 1, 3, -2]$
7	$[1^3, 6, 6, 1, 3, 5, -3]$	11	$[1^2, 2, 1, 1, 1, 2, 3, 2, 2, 1, -2]$
7	$[1^3, 6, 6, 1, 4, 5, -1]$	11	$[1^2, 2, 1, 1, 1, 3, 3, 3, 1, 3, -2]$
7	$[1^3, 6, 6, 1, 5, 2, -5]$	11	$[1^2, 2, 1, 1, 2, 1, 1, 2, 2, 1, -1]$
7	$[1^3, 6, 6, 1, 5, 4, -2]$	11	$[1^2, 2, 1, 1, 2, 2, 1, 3, 1, 1, -1]$
7	$[1^3, 6, 6, 1, 6, 1, -6]$	11	$[1^2, 2, 1, 1, 2, 3, 1, 1, 1, 1, -1]$
7	$[1^3, 6, 6, 1, 6, 3, -2]$	11	$[1^2, 2, 1, 1, 2, 3, 1, 1, 2, 3, -2]$
7	$[1^3, 6, 6, 1, 6, 4, -3]$	11	$[1^2, 2, 1, 1, 3, 2, 2, 1, 2, 1, -3]$
7	$[1^3, 6, 6, 1, 6, 4, -4]$	11	$[1^2, 2, 1, 2, 1, 2, 1, 1, 1, 3, -1]$
8	$[1^2, 2, 1, 1, 1, 2, 1, -2]$	11	$[1^2, 2, 1, 2, 1, 2, 1, 3, 2, 1, -1]$
8	$[1^2, 2, 1, 1, 3, 3, 1, -2]$	11	$[1^2, 2, 1, 2, 1, 3, 1, 1, 1, 1, -1]$
8	$[1^2, 2, 1, 2, 1, 3, 3, -3]$	12	$[1^2, 2, 1, 1, 1, 2, 1, 1, 2, 2, 3, -1]$
9	$[1^2, 2, 1, 1, 1, 2, 1, 3, -3]$	12	$[1^2, 2, 1, 1, 1, 2, 1, 2, 2, 1, 3, -3]$
9	$[1^2, 2, 1, 1, 2, 3, 1, 2, -3]$	12	$[1^2, 2, 1, 1, 1, 2, 2, 1, 1, 3, 1, -1]$
9	$[1^2, 2, 1, 2, 1, 2, 3, 1, -3]$	12	$[1^2, 2, 1, 1, 1, 2, 2, 1, 2, 3, 1, -1]$
9	$[1^2, 2, 1, 2, 1, 3, 3, 1, -1]$	12	$[1^2, 2, 1, 1, 3, 1, 1, 3, 1, 1, 1, -3]$
10	$[1^2, 2, 1, 1, 1, 1, 2, 3, 2, -3]$	12	$[1^2, 2, 1, 1, 3, 1, 1, 3, 1, 1, 3, -3]$
10	$[1^2, 2, 1, 1, 1, 1, 3, 1, 3, -2]$	12	$[1^2, 2, 1, 1, 3, 1, 2, 2, 1, 1, 2, -1]$
10	$[1^2, 2, 1, 1, 1, 1, 3, 3, 1, -3]$	13	$[1^2, 2, 1, 1, 1, 3, 3, 2, 2, 3, 3, 2, -2]$
10	$[1^2, 2, 1, 1, 1, 2, 2, 1, 1, -1]$	13	$[1^2, 2, 1, 2, 1, 1, 1, 1, 1, 2, 3, 2, -1]$
10	$[1^2, 2, 1, 1, 1, 2, 3, 1, 1, -1]$	13	$[1^2, 2, 1, 2, 1, 1, 1, 1, 1, 3, 1, 1, -2]$
10	$[1^2, 2, 1, 1, 1, 3, 2, 1, 3, -3]$	13	$[1^2, 2, 1, 2, 1, 1, 1, 1, 2, 1, 1, 2, -1]$
10	$[1^2, 2, 1, 1, 2, 1, 1, 1, 2, -2]$	13	$[1^2, 2, 1, 2, 1, 1, 1, 1, 2, 1, 3, 1, -2]$
10	$[1^2, 2, 1, 1, 2, 1, 1, 1, 3, -1]$	13	$[1^2, 2, 1, 2, 1, 1, 1, 1, 2, 1, 3, 1, -2]$
10	$[1^2, 2, 1, 1, 2, 1, 2, 3, 1, -3]$	13	$[1^2, 2, 1, 2, 1, 1, 1, 2, 3, 1, 2, 1, -3]$
10	$[1^2, 2, 1, 1, 2, 3, 3, 1, 3, -2]$	13	$[1^2, 2, 1, 2, 1, 1, 1, 3, 1, 3, 2, 3, -2]$
10	$[1^2, 2, 1, 1, 2, 3, 3, 3, 1, -2]$	13	$[1^2, 2, 1, 2, 1, 1, 1, 3, 1, 3, 3, 2, -3]$
10	$[1^2, 2, 1, 1, 3, 2, 2, 1, 3, -1]$	13	$[1^2, 2, 1, 2, 1, 1, 1, 3, 2, 2, 1, 1, -1]$
10	$[1^2, 2, 1, 1, 3, 2, 2, 2, 1, -2]$	13	$[1^2, 2, 1, 2, 1, 1, 1, 3, 3, 3, 3, 1, -1]$
10	$[1^2, 2, 1, 1, 3, 2, 3, 1, 2, -2]$	13	$[1^2, 2, 1, 2, 1, 1, 2, 3, 1, 2, 1, 1, -1]$
10	$[1^2, 2, 1, 1, 3, 3, 3, 2, 3, -2]$	13	$[1^2, 2, 1, 2, 1, 1, 3, 1, 1, 1, 3, 1, -1]$
10	$[1^2, 2, 1, 2, 1, 2, 2, 3, 2, -2]$	13	$[1^2, 2, 1, 2, 1, 1, 3, 1, 3, 3, 1, 2, -1]$
10	$[1^2, 2, 1, 2, 1, 2, 2, 3, 2, -3]$	13	$[1^2, 2, 1, 2, 1, 1, 3, 3, 1, 1, 1, 3, -3]$
10	$[1^2, 2, 1, 2, 1, 3, 1, 3, 3, -3]$	13	$[1^2 2, 1, 2, 1, 1, 3, 3, 2, 2, 1, 3, -1]$
10	$[1^2, 2, 1, 2, 1, 3, 2, 3, 2, -1]$	13	$[1^2, 2, 1, 2, 1, 1, 3, 3, 2, 2, 1, 3, -2]$
10	$[1^2, 2, 1, 2, 1, 3, 2, 3, 2, -2]$	14	$[1^2, 2, 1, 2, 11, 1, 3, 3, 2, 3, 1, 2, -2]$
10	$[1^2, 2, 1, 2, 1, 3, 3, 2, 2, -3]$	15	$[1^2, 2, 1, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, -2]$
10	$[1^2, 2, 1, 2, 2, 3, 1, 1, 3, -1]$	16	$[1^2, 2, 1, 2, 1, 1, 1, 1, 1, 2, 1, 3, 3, 1, 2, -1]$

Table 4: Other PWN found with our search algorithm. Since these numbers are large, only the index sequence is shown. As an example, the first two entries are 54 and 37 digits long, while the last three entries are 3608, 7392 and 14712 digits long respectively. For each ω, entries are in lexicographic order.

Ω	factored weird number	index sequence
7	$2^2 \cdot 13^2 \cdot 19 \cdot 383 \cdot 23203$ †	$[1^2, 2^2, 1, 2, -1]$
7	$2^2 \cdot 13 \cdot 17 \cdot 443^2 \cdot 194867$	$[1^2, 2, 1, 1^2, -6]$
7	$2 \cdot 5^2 \cdot 29 \cdot 37 \cdot 137 \cdot 211$ †	$[1, 1^2, 4, 3, 11, -1]$
7	$2 \cdot 5 \cdot 11^2 \cdot 103 \cdot 877 \cdot 2376097$	$[1, 1, 1^2, 3, 1, -1]$
7	$2 \cdot 5 \cdot 11 \cdot 127^2 \cdot 167 \cdot 223$ †	$[1, 1, 1, 15^2, 15, -1]$
8	$2^3 \cdot 17^2 \cdot 277 \cdot 1979 \cdot 115259$	$[1^3, 1^2, 6, 4, -1]$
8	$2^3 \cdot 23^2 \cdot 53 \cdot 691 \cdot 32587$	$[1^3, 3^2, 1, 3, -1]$
8	$2^2 \cdot 13 \cdot 17 \cdot 449 \cdot 24809^2 \cdot 351659387$	$[1^2, 2, 1, 2, 1^2, -3]$
8	$2^2 \cdot 13 \cdot 17 \cdot 449 \cdot 24809^2 \cdot 351659377$	$[1^2, 2, 1, 2, 1^2, -4]$
9	$2^6 \cdot 137^2 \cdot 1931$ †	$[1^6, 2^2, -1]$
9	$2^4 \cdot 37 \cdot 197 \cdot 58313^2 \cdot 3400230989$	$[1^4, 1, 1, 1^2, -4]$
9	$2^4 \cdot 41 \cdot 131 \cdot 21517^2 \cdot 14007547$	$[1^4, 2, 1, 6^2, -1]$
9	$2^2 \cdot 13 \cdot 17 \cdot 443 \cdot 97919 \cdot 563915543^2 \cdot P_{17}$	$[1^2, 2, 1, 1, 1, 1^2, -5]$
9	$2^2 \cdot 13 \cdot 17 \cdot 449 \cdot 24809^2 \cdot 351659531 \cdot P_{16}$	$[1^2, 2, 1, 2, 1^2, 3, -1]$
10	$2^4 \cdot 41 \cdot 131 \cdot 21493 \cdot 46175611^2 \cdot P_{14}$	$[1^4, 2, 1, 3, 2^2, -5]$
10	$2^3 \cdot 37 \cdot 47 \cdot 59 \cdot 102607 \cdot 1503940237^2 \cdot P_{17}$	$[1^3, 6, 6, 1, 1, 5^2, -6]$
11	$2^8 \cdot 797^2 \cdot 1429$ †	$[1^8, 42^2, -1]$
12	$2^7 \cdot 359 \cdot 883 \cdot 2535977^2 \cdot 6431171736581$	$[1^7, 18, 1, 1^2, -1]$
13	$2^{10} \cdot 2081^2 \cdot 129083$	$[1^{10}, 4^2, -1]$
15	$2^{12} \cdot 9103^2 \cdot 81847$	$[1^{12}, 101^2, -2]$

Table 5: Some of the PWN with square odd prime factors that we have found. PWN already listed in A273815 are marked with †. For $\Omega = 7$, this is the complete list of all the PWN with at least one odd prime factor with exponent greater than one. P_i denotes a generic prime with i digits. Entries are in lexicographic order of index sequences.

Ω	factored weird number	index sequence
12	$w_6 \cdot 13437522702621389^2 \cdot P_{31}^2 \cdot P_{60}$	$[1^2, 2, 1, 2, 1, 1, 1^2, 1^2, -1]$
12	$w_6 \cdot 13437522702621427^2 \cdot P_{31}^2 \cdot P_{60}$	$[1^2, 2, 1, 2, 1, 1, 2^2, 1^2, -3]$
12	$w_6' \cdot 13826118575254057^2 \cdot P_{32}^2 \cdot P_{61}$	$[1^2, 2, 1, 1, 1, 1, 1^2, 1^2, -4]$
13	$w_6' \cdot 13826118575254057^2 \cdot P_{32} \cdot P_{61}^2 \cdot P_{118}$	$[1^2, 2, 1, 1, 1, 1, 1^2, 1, 4^2, -1]$
14	$w_6' \cdot 13826118575254057^2 \cdot P_{32} \cdot P_{60}^2 \cdot P_{118} \cdot P_{233}$	$[1^2, 2, 1, 1, 1, 1, 1^2, 2, 1^2, 2, -1]$
15	$w_6' \cdot 13826118575254057 \cdot P_{32}^2 \cdot P_{61}^2 \cdot P_{120}^2 \cdot P_{237}$	$[1^2, 2, 1, 1, 1, 1, 1, 1^2, 1^2, 2^2, -1]$

Table 6: Some of the PWN with 2 and 3 square odd prime factors that we have found. Here, $w_6 = 2^2 \cdot 13 \cdot 17 \cdot 449 \cdot 24809 \cdot 223797481$, $w_6' = 2^2 \cdot 13 \cdot 17 \cdot 443 \cdot 97919 \cdot 563915543$ and P_i denotes a generic prime with i digits. Entries are in lexicographic order of index sequences.