TAIL BEHAVIORS OF A CLASS OF GENERALIZED WEIGHTED KOLMOGOROV-SMIRNOV STATISTICS

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Abstract: In this paper, we analyze a multivariate counterpart of the generalized weighted Kolmogorov-Smirnov statistic, which is the supremum of weighted locally stationary chi-square process over non-compact interval. The boundedness and the exact tail asymptotic behavior of the statistics are derived. We illustrate our findings by two examples where the statistic is defined by Brownian bridge and fractional Brownian motion respectively.

Key Words: generalized weighted Kolmogorov-Smirnov statistic; chi-square process; exact asymptotics; Brownian bridge; fractional Brownian motion

AMS Classification: Primary 60G15; secondary 60G70

1. INTRODUCTION

Let $X(t), t \geq 0$, be a Gaussian process with almost surely (a.s.) continuous sample paths. For a sequence of constants b_i's satisfying

$$1 = b_1 = \cdots = b_k > b_{k+1} \geq \cdots \geq b_n > 0$$

we define the chi-square process as

$$(1) \qquad \chi^2_{\boldsymbol{b}}(t) = \sum_{i=1}^{n} b_i^2 X_i^2(t), \quad t \geq 0,$$

where X_i's are independent copies of X. The supremum of chi-square process appears naturally as limiting test statistic in various statistical models; see, e.g., [1, 2, 3, 4]. It also plays an important role in reliability applications in the engineering sciences, see [5, 6, 7, 8] and the references therein.

Of interest in applied probability and statistics is the tail distribution of $\sup_{t \in \mathcal{T}} \chi^2_{\boldsymbol{b}}(t)$ with $\mathcal{T} \subset \mathbb{R}_+$, provided that

$$(2) \qquad \sup_{t \in \mathcal{T}} \chi^2_{\boldsymbol{b}}(t) < \infty \quad a.s..$$

Numerous contributions have been devoted to the study of the tail asymptotics of the supremum of chi-square processes and Gaussian chaos processes, an extension of chi-square processes, over compact intervals $\mathcal{T}$; see, e.g., [6, 9, 10, 11, 12], where the technique used for chi-square processes is to transform the supremum of chi-square process into the supremum of a special Gaussian random field by resorting to the duality of norms. We refer to, e.g., [13, 14, 15, 16, 17, 18, 19, 20] for more discussions on the tail asymptotics (or excursion probability) of supremum of Gaussian and related fields.

In statistical applications, it is common that $\mathcal{T}$ is a non-compact interval, namely,

$$\mathcal{T} = (0, 1) \text{ or } (0, 1].$$

However,

$$\sup_{t \in \mathcal{T}} \chi_{\boldsymbol{b}}^2(t) = \infty \quad a.s. \tag{3}$$

is possible. For instance, in [21, 22] a special case of $\chi_{\boldsymbol{b}}$ with $n = 1$ is considered, i.e.,

$$\chi_{\boldsymbol{b}}(t) = \left| \overline{B}(t) \right| = \frac{|B(t)|}{\sqrt{t(1-t)}}, \quad t \in (0,1),$$

where B is the standard Brownian bridge with variance function $Var(B(t)) = t(1-t), t \in [0,1]$. By the law of iterated logarithm of Brownian motion, we have

$$\sup_{t \in (0,1)} \left| \overline{B}(t) \right| = \infty \quad a.s..$$

To overcome this unboundedness, researchers have focused on the following generalized weighted Kolmogorov-Smirnov statistic

$$W_w := \sup_{t \in (0,1)} \frac{\left| \overline{B}(t) \right|}{w(t)}$$

with a suitably chosen weight function w so that $W_w < \infty$ a.s., see, e.g., [23]. It is noted in [23][Theorem 3.3, Theorem 4.2.3] (see also [24][Theorem 26.3]) that

$$W_w < \infty \quad a.s. \quad \Leftrightarrow \quad \int_0^1 \frac{1}{t(1-t)} e^{-cw^2(t)} dt < \infty \text{ for some } c > 0. \tag{4}$$

As discussed in [25], the weight function is introduced when the Goodness-of-Fit test is intended to emphasize a specific region of the domain. Moreover, it is also shown in [24] that for specific types of problems, the weighted versions provide greater power than the original unweighted versions, provided that the weighting function is properly chosen. We refer to [23, 24, 26, 27] for further discussions on the generalized weighted Kolmogorov-Smirnov statistic.

In this paper, we are interested in the analysis of a class of multivariate counterparts of the generalized weighted Kolmogorov-Smirnov statistics, namely, the supremum of weighted chi-square processes defined by

$$\sup_{t \in \mathcal{T}} \frac{\chi_{\boldsymbol{b}}^2(t)}{w^2(t)}, \quad \text{with } \mathcal{T} = (0,1) \text{ or } (0,1],$$

where w is some positive continuous function defined on $\mathcal{T}$, and the generic process X is the locally stationary Gaussian process. More precisely, $X(t), t \in \mathcal{T}$, is a centered Gaussian process with a.s. continuous sample paths, unit variance and correlation function $r(\cdot, \cdot)$ satisfying

$$\lim_{h \to 0} \frac{1 - r(t, t+h)}{K^2(|h|)} = C(t) \tag{5}$$

uniformly in $t \in I$, for all the compact interval I in $\mathcal{T}$, where $K(\cdot)$ is a positive regularly varying function at 0 with index $\alpha/2 \in (0,1]$, and $C(\cdot)$ is a positive continuous function satisfying

$$\lim_{t \to 0} C(t) = \infty \quad \text{or} \quad \lim_{t \to 1} C(t) = \infty.$$

We refer to [28] for the introduction of locally stationary Gaussian processes and [29, 30, 31] for discussions on the general $\alpha(t)$-locally stationary Gaussian processes.

As an extension of (4), we show in Theorem 3.1 necessary and sufficient conditions on the weight function w under which it holds that

$$\text{(6)} \qquad \sup_{t \in \mathcal{T}} \frac{\chi_{\boldsymbol{b}}^2(t)}{w^2(t)} < \infty \qquad \text{a.s..}$$

Furthermore, for certain w satisfying (6) we derive in Theorem 3.3 the exact asymptotics of

$$\text{(7)} \qquad \mathbb{P}\left\{ \sup_{t \in \mathcal{T}} \frac{\chi_{\boldsymbol{b}}^2(t)}{w^2(t)} > u \right\}, \qquad u \to \infty.$$

Due to wide applications of Kolmogorov-Smirnov statistics and their variations in statistics, we expect that the derived results will have interesting statistical applications.

We note in passing that there exist alternative approaches to overcome the problem of unboundedness. For example, in [1, 2, 4] the so-called "trimmed supremum-type" test statistics are introduced and in [32, 33] the chi-square processes with trend are discussed.

Organization of the rest of the paper: In Section 2 we present a preliminary result which is an adaption of Theorem A.1 in [32]. The main results are given in Section 3, followed by some examples. Finally, all the proofs are displayed in Section 4.

2. PRELIMINARIES

This section concerns a result derived in [32], which is crucial for the derivation of (6). Based on the discussions therein, we shall consider $\int_0^{1/2} (C(s))^{1/\alpha} ds = \infty$ or $\int_{1/2}^1 (C(s))^{1/\alpha} ds = \infty$. For this purpose, of crucial importance is the following function

$$f(t) = \int_{1/2}^t (C(s))^{1/\alpha} ds, \quad t \in (0,1).$$

We denote by $\overleftarrow{f}(t), t \in (f(0), f(1))$ the inverse function of $f(t), t \in (0,1)$. Further, for any $d > 0$, let $s_{j,d}^{(1)} = \overleftarrow{f}(jd)$, $j \in \mathbb{N} \cup \{0\}$ if $f(1) = \infty$, and let $s_{j,d}^{(0)} = \overleftarrow{f}(-jd)$, $j \in \mathbb{N} \cup \{0\}$ if $f(0) = -\infty$. Denote $\Delta_{j,d}^{(1)} = [s_{j-1,d}^{(1)}, s_{j,d}^{(1)}], j \in \mathbb{N}$ and $\Delta_{j,d}^{(0)} = [s_{j,d}^{(0)}, s_{j-1,d}^{(0)}], j \in \mathbb{N}$, which give a partition of $[1/2, 1)$ in the case $f(1) = \infty$ and a partition of $(0, 1/2]$ in the case $f(0) = -\infty$, respectively. Moreover, let $q(u) = \overleftarrow{K}(u^{-1/2})$ be the inverse function of $K(\cdot)$ at point $u^{-1/2}$ (assumed to exist asymptotically).

The following (scenario-dependent) restrictions on the positive continuous weight function w^2 and the correlation function $r(\cdot, \cdot)$ of X play a crucial role. Let therefore $S \in \{0, 1\}$.

Condition A(S): The weight function w^2 is monotone in a neighborhood of S and satisfies $\lim_{t \to \infty} w^2(t) = \infty$.

Condition B(S): Suppose that there exists some constant $d_0 > 0$ such that

$$\limsup_{j \to \infty} \sup_{t \neq s \in \Delta_{j,d_0}^{(S)}} \frac{1 - r(t,s)}{K^2(|f(t) - f(s)|)} < \infty,$$

and when $\alpha = 2$ and $k = 1$, assume further

$$K^2(|t|) = O(t^2), \quad t \to 0.$$

Condition C(S): Suppose that there exists some constant $d_0 > 0$ such that

$$\liminf_{j \to \infty} \inf_{t \neq s \in \Delta_{j,d_0}^{(S)}} \frac{1 - r(t,s)}{K^2(|f(t) - f(s)|)} > 0.$$

Moreover, there exist $j_0, l_0 \in \mathbb{N}$, $M_0, \beta > 0$, such that for $j \geq j_0$, $l \geq l_0$,

(8)
$$\sup_{s \in \Delta_{j+l,d_0}^{(S)}, t \in \Delta_{j,d_0}^{(S)}} |r(s,t)| < M_0 l^{-\beta}.$$

For the subsequent discussions we present Theorem A.1 of [32], focusing on $|f(S)| = \infty$. Recalling that $q(u) = \overleftarrow{K}(u^{-1/2})$ is the inverse function of $K(\cdot)$ at point $u^{-1/2}$, we define

$$I_w(S) = \left| \int_{1/2}^{S} (C(t))^{1/\alpha} \frac{(w(t))^{k-2}}{q(w^2(t))} e^{-\frac{w^2(t)}{2}} dt \right|.$$

Theorem 2.1. *Let $X(t), t \in (0,1)$, be a centered locally stationary Gaussian process with a.s. continuous sample paths, unit variance and correlation function $r(\cdot, \cdot)$ satisfying (5) and $r(s,t) < 1$ for $s \neq t \in (0,1)$. Suppose further that, for $S = 0$ or 1, we have $|f(S)| = \infty$ and **A**(S), **B**(S), **C**(S) are satisfied. Then*

(9)
$$\mathbb{P}\left\{ \chi_{\boldsymbol{b}}^2(t) \leq w^2(t) \quad \text{ultimately as} \ \ t \to S \right\} = 0, \quad or \quad 1$$

according to

$$I_w(S) = \infty, \quad or \quad < \infty.$$

3. Main Results

In this section, we first give a criteria for (6) to hold and then display the exact asymptotics of (7) for different types of w such that (6) is valid.

3.1. **Analysis of** (6). Denote by $E(0) = (0, 1/2]$ and $E(1) = [1/2, 1)$. Recall that $S \in \{0, 1\}$. Under the conditions of Theorem 2.1, we have that if $I_w(S) < \infty$, then

$$\sup_{t \in E(S)} \frac{\chi_{\boldsymbol{b}}^2(t)}{w^2(t)} < \infty \qquad \text{a.s.,}$$

however, when $I_w(S) = \infty$ we only see that

$$\sup_{t \in E(S)} \frac{\chi_{\boldsymbol{b}}^2(t)}{w^2(t)} \geq 1 \qquad \text{a.s..}$$

Clearly, the above is not informative for the validity of (6). On the other hand, it is easily shown that

$$\sup_{t \in E(S)} \frac{\chi_{\boldsymbol{b}}^2(t)}{w^2(t)} < \infty \ \ a.s. \quad \Leftrightarrow \quad \sup_{t \in E(S)} \frac{|X(t)|}{w(t)} < \infty \ \ a.s.,$$

which means that, instead of the condition $I_w(S) = \infty$ in Theorem 2.1, a more accurate condition that is independent of n, k should be possible to ensure that (6) holds. Inspired by this fact and given the importance of (6), we provide below a sufficient and necessary condition for (6) to hold.

Define, for any constant $c > 0$ and any positive continuous function w

$$J_{c,w}(S) = \left| \int_{1/2}^{S} (C(t))^{1/\alpha} e^{-cw^2(t)} dt \right|.$$

Below is our first principal result, a criterion for (6), which is a generalization of (4).

Theorem 3.1. *Under the conditions of Theorem 2.1 we have*

$$\sup_{t \in E(S)} \frac{\chi_{\boldsymbol{b}}^2(t)}{w^2(t)} < \infty \ \ a.s. \quad \Leftrightarrow \quad J_{c,w}(S) < \infty \quad \text{for some} \ \ c > 0.$$

Next we illustrate the criteria presented in Theorem 3.1 by an example of a weighted chi-square process with generic process being the normalized standard Brownian bridge, which further provides us with a clear comparison between $I_w(S)$ and $J_{c,w}(S)$.

Example 3.2. *Let $X(t) = \overline{B}(t), t \in (0,1)$, and, with $\rho_1 > 0, \rho_2 \in \mathbb{R}$, define*

$$(10) \qquad w^2_{\rho_1,\rho_2}(t) = 2\rho_1 \ln\ln\left(\frac{e^2}{t(1-t)}\right) + 2\rho_2 \ln\ln\ln\left(\frac{e^2}{t(1-t)}\right), \quad t \in (0,1).$$

First note that for the normalized standard Brownian bridge

$$(11) \qquad \lim_{h \to 0} \frac{1 - \mathbb{E}\left(\overline{B}(t)\overline{B}(t+h)\right)}{|h|} = \frac{1}{2t(1-t)}$$

holds uniformly in $t \in I$, for any compact interval I in $(0,1)$. This means that $\overline{B}$ is a locally stationary Gaussian process with

$$K(h) = \sqrt{|h|}, \quad \alpha = 1, \quad q(u) = u^{-1}.$$

Furthermore,

$$f(t) = \int_{1/2}^t \frac{1}{2s(1-s)} ds = \frac{1}{2} \ln\left(\frac{t}{1-t}\right)$$

implying that $f(1) = -f(0) = \infty$. Moreover, by the proof of Corollary 2.6 in [32] we have that conditions **B(S)** *and* **C(S)** *are satisfied by $\overline{B}(t), t \in (0,1)$, and $\mathbb{E}\left(\overline{B}(t), \overline{B}(s)\right) < 1$ for $s \neq t, s, t \in (0,1)$. Thus, all the conditions of Theorem 3.1 are fulfilled.*

Next, on one hand, we have

$$\frac{1}{t(1-t)}(w_{\rho_1,\rho_2}(t))^k e^{-\frac{w^2_{\rho_1,\rho_2}(t)}{2}} \sim \frac{Q}{t(1-t)\left(\ln\left(\frac{1}{t(1-t)}\right)\right)^{\rho_1}\left(\ln\ln\left(\frac{e^2}{t(1-t)}\right)\right)^{\rho_2-k/2}}$$

as $t \to 0$ or $t \to 1$, with Q some positive constant. Thus, elementary calculations show that

$$I_w(0) = I_w(1) = \int_{1/2}^1 \frac{(w_{\rho_1,\rho_2}(t))^k}{t(1-t)} e^{-\frac{w^2_{\rho_1,\rho_2}(t)}{2}} dt < \infty$$

holds if and only if

$$(12) \qquad \rho_1 > 1, \quad \text{or } \rho_1 = 1 \text{ and } \rho_2 > 1 + k/2.$$

On the other hand, we can show that the functions $w_{\rho_1,\rho_2}(t)$ satisfying that $\exists c > 0$ such that $J_{c,w}(S) < \infty$ are not restricted to the ones satisfying (12). In fact, since for any $\rho_1 > 0$ there exists some c such that $\rho_1 > \frac{1}{2c}$, we have that

$$\begin{aligned} J_{c,w}(0) = J_{c,w}(1) &= \int_{1/2}^1 \frac{1}{t(1-t)} e^{-cw^2_{\rho_1,\rho_2}(t)} dt \\ &\leq \int_{1/2}^1 \frac{1}{t(1-t)\left(\ln\left(\frac{1}{t(1-t)}\right)\right)^{2c\rho_1}\left(\ln\ln\left(\frac{e^2}{t(1-t)}\right)\right)^{2c\rho_2}} dt < \infty \end{aligned}$$

holds for any $\rho_2 \in \mathbb{R}$. Thus, we conclude from Theorem 3.1 that

$$\sup_{t \in (0,1)} \frac{\chi^2_{\boldsymbol{b}}(t)}{w^2_{\rho_1,\rho_2}(t)} < \infty \quad a.s.$$

holds for any $\rho_1 > 0$ and $\rho_2 \in \mathbb{R}$.

The exact tail asymptotics of $\sup_{t \in (0,1)} \frac{\chi^2_{\boldsymbol{b}}(t)}{w^2_{\rho_1,\rho_2}(t)}$ will be discussed in next section.

3.2. **Asymptotics of** (7). For those w such that (6) holds, of interest is the exact tail asymptotic behavior of $\sup_{t \in \mathcal{T}} \frac{\chi_b^2(t)}{w^2(t)}$. Actually, as we have seen, the behavior of w around 0 and 1 plays a crucial role for the finiteness in (6). However, this does not apply to the tail asymptotics of $\sup_{t \in \mathcal{T}} \frac{\chi_b^2(t)}{w^2(t)}$. It turns out that only the probability mass in the neighborhood of minimizer of w contribute to the tail asymptotics, indicating that the other part of the process including the part around 0 or 1 can be neglected. As discussed in the Introduction, the weight function is sometimes introduced when the Goodness-of-Fit test is intended to emphasize a specific region of the domain. Motivated by this, for the tail asymptotics we shall consider the following two types of w:

Assumption F1: The function w attains its minimum at finite distinct inner points $t_i, i = 1, \ldots, m$ of $\mathcal{T}$, and

$$(13) \qquad\qquad w(t_i + t) = w(t_i) + a_i \, |t_i|^{\beta_i} \, (1 + o(1)), \quad t \to t_i$$

holds for some positive constants $a_i, \beta_i > 0, i = 1, \ldots, m$.

Assumption F2: The function w attains its minimum on disjoint intervals $[c_i, d_i] \subseteq \mathcal{T}$, $i = 1, \ldots, m$. Under assumption **F1**, we need additional conditions which are stated below. Recall $q(u) = \overleftarrow{K}(u^{-1/2})$. It follows that $q(u)$ is a regularly varying function at infinity with index $-1/\alpha$ which can be further expressed as $q(u) = u^{-1/\alpha} L(u^{-1/2})$, with $L(\cdot)$ a slowly varying function at 0. Denote further $\beta = \max_{1 \le i \le m} \beta_i$. According to the values of $L(u^{-1/2})$ as $u \to \infty$, we consider the following three scenarios:

C1(β): $\beta > \alpha$, or $\beta = \alpha$ and $\lim_{u \to \infty} L(u^{-1/2}) = 0$;

C2(β): $\beta = \alpha$ and $\lim_{u \to \infty} L(u^{-1/2}) = \mathcal{L} \in (0, \infty)$;

C3(β): $\beta < \alpha$, or $\beta = \alpha$ and $\lim_{u \to \infty} L(u^{-1/2}) = \infty$.

Before displaying our results, we introduce two important constants. One is the *Pickands constant* defined by

$$\mathcal{H}_{2H} = \lim_{S \to \infty} \frac{1}{S} \mathbb{E} \left(\exp \left(\sup_{t \in [0, S]} \left(\sqrt{2} B_H(t) - t^\alpha \right) \right) \right),$$

with $B_H(t), t \in \mathbb{R}$, a standard fractional Brownian motion (fBm) defined on $\mathbb{R}$ with Hurst index $H \in (0, 1]$. And the other one is the *Piterbarg constant* defined by

$$\mathcal{P}_{2H}^d = \lim_{\lambda \to \infty} \mathbb{E} \left(\exp \left(\sup_{t \in [-\lambda, \lambda]} \left(\sqrt{2} B_H(t) - (1 + d) \, |t|^\alpha \right) \right) \right), \quad d > 0.$$

We refer to [14, 34, 35, 36, 37, 38] for the properties and generalizations of the Pickands-Piterbarg type constants. Moreover, We shall use the standard notation for asymptotic equivalence of two functions f and h. Specifically, we write $f(x) \sim h(x)$, if $\lim_{x \to a} f(x)/h(x) = 1$ ($a \in \mathbb{R} \cup \{\infty\}$), and further, write $f(x) = o(h(x))$, if $\lim_{x \to a} f(x)/h(x) = 0$.

Let $V = \{1 \le i \le m : \beta_i = \beta\}$ and $V^c = \{1 \le i \le m : \beta_i < \beta\}$. Below is our second principal result.

Theorem 3.3. *Let* $\frac{\chi_b^2(t)}{w^2(t)}, t \in \mathcal{T}$, *be the weighted locally stationary chi-square process considered in Theorem 2.1 such that* (6) *holds. We have:*

(i). If **F1** *is satisfied, then, as* $u \to \infty$,

$$\mathbb{P} \left\{ \sup_{t \in \mathcal{T}} \frac{\chi_b^2(t)}{w^2(t)} > u \right\} \sim \left(\prod_{i=k+1}^n (1 - b_i^2)^{-1/2} \right) \mathcal{M}(u) \; \Upsilon_k(w^2(t_1)u),$$

where (with the convention $\prod_{i=p}^{q} = 1$ if $q < p$)

$$(14) \qquad \Upsilon_k(u) := \mathbb{P}\left\{\chi_{k,\mathbf{1}}^2(0) > u\right\} = \frac{2^{(2-k)/2}}{\Gamma(k/2)} u^{k/2-1} \exp\left(-\frac{u}{2}\right), \quad u > 0,$$

and

$$\mathcal{M}(u) = \begin{cases} 2\left(\sum_{i \in V} a_i^{-1/\beta}(C(t_j))^{1/\alpha}\right)(w(t_1))^{2/\alpha - 1/\beta}\Gamma(1/\beta + 1)\mathcal{H}_\alpha(q(u))^{-1}u^{-1/\beta}, & \text{for } \mathbf{C1(\beta)}, \\ \sum_{i \in V} \mathcal{P}_\alpha^{a_i(w(t_1)C(t_j))^{-1}\mathcal{L}^\alpha} + \sharp V^c, & \text{for } \mathbf{C2(\beta)}, \\ m, & \text{for } \mathbf{C3(\beta)}. \end{cases}$$

(ii). If **F2** *is satisfied, then, as* $u \to \infty$,

$$\mathbb{P}\left\{\sup_{t \in \mathcal{T}} \frac{\chi_{\mathbf{b}}^2(t)}{w^2(t)} > u\right\} \sim \left(\prod_{i=k+1}^{n}\left(1 - b_i^2\right)^{-1/2}\right)\left(\sum_{j=1}^{m}\int_{c_j}^{d_j}(C(t))^{1/\alpha}dt\right)\mathcal{H}_\alpha\left(q(w^2(c_1)u)\right)^{-1}\Upsilon_k(w^2(c_1)u).$$

We conclude this section with two applications of Theorem 3.3. The first one is on the weighted locally stationary chi-square process discussed in Example 3.2, and the second one concerns the weighted locally stationary chi-square process with generic process X being a normalized standard fBm.

Corollary 3.4. *Let* $\frac{\chi_{\mathbf{b}}^2(t)}{w_{\rho_1,\rho_2}^2(t)}, t \in (0,1)$, *with* $\rho_1 > 0$ *and* $\rho_2 \in \mathbb{R}$, *be the weighted locally stationary chi-square process discussed in Example 3.2. We have, as* $u \to \infty$, *if* $\rho_2 \geq -\rho_1 \ln\ln(4e^2)$, *then*

$$\mathbb{P}\left\{\sup_{t \in (0,1)} \frac{\chi_{\mathbf{b}}^2(t)}{w_{\rho_1,\rho_2}^2(t)} > u\right\} \sim \left(\prod_{i=k+1}^{n}(1 - b_i^2)^{-1/2}\right)\mathcal{M}(u)\Upsilon_k(2A_1 u),$$

where $A_1 = \rho_1 \ln\ln\left(4e^2\right) + \rho_2 \ln\ln\ln\left(4e^2\right)$ *and*

$$\mathcal{M}(u) = \begin{cases} 2A_1\sqrt{\frac{\pi \ln(4e^2)\ln\ln(4e^2)}{\rho_1 \ln\ln(4e^2)+\rho_2}}u^{1/2}, & \text{for } \rho_2 > -\rho_1 \ln\ln(4e^2), \\ 2\Gamma(1/4)A_1\left(\frac{\ln\ln\left(4e^2\right)\left(\ln(4e^2)\right)^2}{8\rho_1}\right)^{1/4}u^{3/4}, & \text{for } \rho_2 = -\rho_1 \ln\ln(4e^2), \end{cases}$$

and if $\rho_2 < -\rho_1 \ln\ln(4e^2)$, *then*

$$\mathbb{P}\left\{\sup_{t \in (0,1)} \frac{\chi_{\mathbf{b}}^2(t)}{w_{\rho_1,\rho_2}^2(t)} > u\right\} \sim 2A_2\left(\prod_{i=k+1}^{n}(1 - b_i^2)^{-1/2}\right)\rho_1^{-1}Q\sqrt{-2\pi\rho_2}u^{1/2}\Upsilon_k(2A_2 u),$$

where $A_2 = \rho_2(\ln(-\rho_2) - \ln(\rho_1) - 1)$ *and*

$$Q = \frac{1}{2t_1 - 1}\ln\left(\frac{e^2}{t_1(1 - t_1)}\right), \quad t_1 = 1/2 + \sqrt{1/4 - e^{2-e^{-\rho_2/\rho_1}}}.$$

Next, we consider $B_H(t), t \geq 0$, to be the standard fBm with Hurst index $H \in (0,1)$ and covariance function

$$Cov(B_H(s), B_H(t)) = \frac{1}{2}\left(|s|^{2H} + |t|^{2H} - |s - t|^{2H}\right), \quad s, t \geq 0.$$

Denote by $\overline{B}_H(t) = B_H(t)/t^H, t \in (0,1]$ the normalized standard fBm defined on $(0,1]$. Further, for any $\rho > 0$ and $\varepsilon \in (0,1)$, we define

$$(15) \qquad w_{\rho,\varepsilon}^2(t) = \begin{cases} \rho \ln\ln\left(e^2/t\right), & \text{for } t \in (0, \varepsilon), \\ \rho \ln\ln\left(e^2/\varepsilon\right), & \text{for } t \in [\varepsilon, 1]. \end{cases}$$

We have the following result.

Corollary 3.5. Let $\frac{\chi_{\boldsymbol{b}}^2(t)}{w_{\rho,\varepsilon}^2(t)}, t \in (0,1]$, be a weighted chi-square process with generic process $\overline{B}_H(t), t \in (0,1]$ and $w_{\rho,\varepsilon}^2$ given in (15). Then, we have, as $u \to \infty$,

$$\mathbb{P}\left\{\sup_{t \in (0,1)} \frac{\chi_{\boldsymbol{b}}^2(t)}{w_{\rho,\varepsilon}^2(t)} > u\right\} \sim \left(\prod_{i=k+1}^n (1 - b_i^2)^{-1/2}\right)(-\ln(\varepsilon))\left(\ln\ln\left(e^2/\varepsilon\right)\rho/2\right)^{\frac{1}{2H}}$$

$$\times \mathcal{H}_{2H} u^{\frac{1}{2H}} \Upsilon_k(\rho \ln\ln\left(e^2/\varepsilon\right)u).$$

4. Proofs

This section is devoted to the proof of all the results presented in Section 3.

Proof of Theorem 3.1: Note that $t^{k/2-1}(q(t))^{-1}$ is a positive regularly varying function at ∞ with index $\kappa = k/2 - 1 + 1/\alpha \geq 0$. Thus, by Potter bound (e.g., [39])

$$c_1 t^{\kappa-1} \leq t^{k/2-1}(q(t))^{-1} \leq c_2 t^{\kappa+1}, \quad t \geq c_3,$$

holds for some constants $c_1, c_2, c_3 > 0$, which, together with the fact that $w^2(t) \to \infty$ as $t \to S$, leads to

$$(16) \qquad Q_1 e^{-w^2(t)} \leq \frac{(w(t))^{k-2}}{q(w^2(t))} e^{-\frac{w^2(t)}{2}} \leq Q_2 e^{-\frac{w^2(t)}{3}}$$

for all t approaching S, with some positive constants Q_1, Q_2. Therefore, if $J_{c,w}(S) < \infty$ holds for some $c > 0$, then, by (16),

$$I_{\sqrt{3cw}}(S) < \infty.$$

This together with Theorem 2.1 yields that

$$\limsup_{t \to S} \frac{\chi_{\boldsymbol{b}}^2(t)}{w^2(t)} \leq 3c \quad a.s.$$

showing that

$$\sup_{t \in E(S)} \frac{\chi_{\boldsymbol{b}}^2(t)}{w^2(t)} < \infty \quad a.s.$$

On the other hand, if $J_{c,w}(S) = \infty$ for all $c > 0$, then, by (16), $I_{\sqrt{cw}}(S) = \infty$. Thus by Theorem 2.1

$$\limsup_{t \to S} \frac{\chi_{\boldsymbol{b}}^2(t)}{w^2(t)} \geq c \quad a.s.$$

holds for all $c > 0$, implying that

$$\sup_{t \in E(S)} \frac{\chi_{\boldsymbol{b}}^2(t)}{w^2(t)} = \infty \quad a.s.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We show next a version of the Borell-TIS inequality for chi-square process, which will play a key role in the proof of Theorem 3.3. We refer to, e.g., [13, 40] for discussions on the Borell-TIS inequality for Gaussian random fields. Denote below $\mathcal{S} \subseteq \mathbb{R}$ to be any fixed interval.

Lemma 4.1. Let $\chi_{\boldsymbol{b}}^2(t), t \in \mathcal{S}$, be a chi-square process with generic centered Gaussian process X which has a.s. continuous sample paths and variance function denoted by $\sigma_X^2(t)$. If

$$\sup_{t \in \mathcal{S}} X(t) < \infty \quad a.s.,$$

then there exists some positive constant Q such that for all $u > Q^2$ we have

$$(17) \qquad \mathbb{P}\left\{\sup_{t \in \mathcal{S}} \chi_{\boldsymbol{b}}^2(t) > u\right\} \leq \exp\left(-\frac{(\sqrt{u} - Q)^2}{2\sup_{t \in \mathcal{S}} \sigma_X^2(t)}\right).$$

Proof of Lemma 4.1: Using the classical approach when dealing with chi-square processes as, e.g., in [4, 14, 32], we introduce a particular Gaussian random field, namely,

$$(18) \qquad Y_{\boldsymbol{b}}(t, \boldsymbol{\theta}) \;=\; \sum_{i=1}^{n} b_i X_i(t) v_i(\boldsymbol{\theta}), \quad (t, \boldsymbol{\theta}) \in \mathcal{D} =: \mathcal{S} \times [-\pi, \pi] \times [-\pi/2, \pi/2]^{n-2},$$

where $\boldsymbol{\theta} = (\theta_2, \theta_3, \cdots, \theta_n)$, and $v_n(\boldsymbol{\theta}) = \sin(\theta_n)$, $v_{n-1}(\boldsymbol{\theta}) = \sin(\theta_{n-1})\cos(\theta_n), \cdots, v_1(\boldsymbol{\theta}) = \cos(\theta_n) \cdots \cos(\theta_2)$ are spherical coordinates. It follows that for any $u > 0$

$$(19) \qquad \mathbb{P}\left(\sup_{t \in \mathcal{S}} \chi_{\boldsymbol{b}}^2(t) > u\right) = \mathbb{P}\left(\sup_{(t, \boldsymbol{\theta}) \in \mathcal{D}} Y_{\boldsymbol{b}}(t, \boldsymbol{\theta}) > \sqrt{u}\right).$$

Since the variance function of $Y_{\boldsymbol{b}}$ satisfies for $u > 0$

$$\mathbb{E}\left((Y_{\boldsymbol{b}}(t, \boldsymbol{\theta}))^2\right) = \sigma_X^2(t)\left(1 - (1 - b_n^2)\sin^2(\theta_n) - \sum_{i=k+1}^{n-1}(1 - b_i^2)\left(\prod_{j=i+1}^{n}\cos^2(\theta_j)\right)\sin^2(\theta_i)\right),$$

we have

$$(20) \qquad \sup_{(t, \boldsymbol{\theta}) \in \mathcal{D}} \mathbb{E}\left((Y_{\boldsymbol{b}}(t, \boldsymbol{\theta}))^2\right) \le \sup_{t \in \mathcal{S}} \sigma_X^2(t).$$

Then, by (19) and the Borell-TIS inequality for Gaussian random fields (cf. [13][Theorem 2.1.1]) we conclude that (17) holds with $Q = \mathbb{E}\left(\sup_{(t, \boldsymbol{\theta}) \in \mathcal{D}} Y_{\boldsymbol{b}}(t, \boldsymbol{\theta})\right) < \infty$. This completes the proof. $\qquad \square$

The next result concerns a upper bound for the tails of double-sup of the locally stationary chi-square processes, which will also play a key role in the proof of Theorem 3.3.

Lemma 4.2. *Let $\chi_{\boldsymbol{b}}^2(t), t \in \mathcal{S}$, be a locally stationary chi-square process with generic centered Gaussian process X which has a.s. continuous sample paths. If further the correlation function of X satisfies*

$$(21) \qquad r(s, t) < 1 \text{ for any } s \neq t \in \mathcal{S},$$

then, for any compact intervals $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{S}$ such that $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$ we have

$$\mathbb{P}\left\{\sup_{t \in \mathcal{S}_1} \chi_{\boldsymbol{b}}^2(t) > u, \; \sup_{t \in \mathcal{S}_2} \chi_{\boldsymbol{b}}^2(t) > u\right\} \le \exp\left(-\frac{(2\sqrt{u} - Q)^2}{2(2 + 2\eta)}\right).$$

for all $u > Q^2$, with some constant $Q > 0$ and $\eta \in (0, 1)$.

Proof of Lemma 4.2: Using the expression of $Y_{\boldsymbol{b}}(t, \boldsymbol{\theta})$ given in (18), we have

$$\mathbb{P}\left\{\sup_{t \in \mathcal{S}_1} \chi_{\boldsymbol{b}}^2(t) > u, \; \sup_{t \in \mathcal{S}_2} \chi_{\boldsymbol{b}}^2(t) > u\right\} = \mathbb{P}\left\{\sup_{(t, \boldsymbol{\theta}) \in \mathcal{D}_1} Y_{\boldsymbol{b}}(t, \boldsymbol{\theta}) > \sqrt{u}, \; \sup_{(t, \boldsymbol{\theta}) \in \mathcal{D}_2} Y_{\boldsymbol{b}}(t, \boldsymbol{\theta}) > \sqrt{u}\right\}$$

$$\le \mathbb{P}\left\{\sup_{(t, \boldsymbol{\theta}) \in \mathcal{D}_1, (t', \boldsymbol{\theta}') \in \mathcal{D}_2} (Y_{\boldsymbol{b}}(t, \boldsymbol{\theta}) + Y_{\boldsymbol{b}}(t', \boldsymbol{\theta}')) > 2\sqrt{u}\right\}$$

where

$$\mathcal{D}_i = \mathcal{S}_i \times [-\pi, \pi] \times [-\pi/2, \pi/2]^{n-2}, \quad i = 1, 2.$$

By (21) we have that there exists some $\eta \in (0,1)$ such that

$$\mathbb{E}\left((Y_{\boldsymbol{b}}(t,\boldsymbol{\theta}) + Y_{\boldsymbol{b}}(t',\boldsymbol{\theta}'))^2\right) = \mathbb{E}\left((Y_{\boldsymbol{b}}(t,\boldsymbol{\theta}))^2\right) + \mathbb{E}\left((Y_{\boldsymbol{b}}(t',\boldsymbol{\theta}'))^2\right) + 2\sum_{i=1}^n \mathbb{E}\left(X_i(t)X_i(t')\right)b_i^2 v_i(\boldsymbol{\theta})v_i(\boldsymbol{\theta}')$$

$$\leq 2 + 2\eta \sum_{i=1}^n b_i^2 v_i(\boldsymbol{\theta})v_i(\boldsymbol{\theta}')$$

$$(22) \qquad \leq 2 + 2\eta, \quad (t,\boldsymbol{\theta}) \in \mathcal{D}_1, (t',\boldsymbol{\theta}') \in \mathcal{D}_2.$$

Consequently, by the Borell-TIS inequality

$$\mathbb{P}\left\{\sup_{(t,\boldsymbol{\theta})\in\mathcal{D}_1,(t',\boldsymbol{\theta}')\in\mathcal{D}_2}(Y_{\boldsymbol{b}}(t,\boldsymbol{\theta}) + Y_{\boldsymbol{b}}(t',\boldsymbol{\theta}')) > 2\sqrt{u}\right\} \leq \exp\left(-\frac{(2\sqrt{u}-Q)^2}{2(2+2\eta)}\right)$$

for all $u > \frac{Q^2}{4}$, with $Q = \mathbb{E}\left(\sup_{(t,\boldsymbol{\theta})\in D_1,(t',\boldsymbol{\theta}')\in D_2}(Y_{\boldsymbol{b}}(t,\boldsymbol{\theta}) + Y_{\boldsymbol{b}}(t',\boldsymbol{\theta}'))\right) < \infty$. Thus, the claim follows. This completes the proof. $\qquad\square$

Proof of Theorem 3.3: Without loss of generality, we show the proof only for the case where $\mathcal{T} = (0,1)$.
(i). Let $\rho > 0$ be a sufficiently small constant such that

$$[t_i - \rho, t_i + \rho] \cap [t_j - \rho, t_j + \rho] = \emptyset, \quad \text{for all } i \neq j.$$

Further, denote $\mathcal{T}_\rho = \mathcal{T} \setminus \bigcup_{i=1}^m [t_i - \rho, t_i + \rho]$. It follows from the Bonferroni inequality (e.g., [41]) that

$$\sum_{i=1}^m p_i(u) + \mathbb{P}\left\{\sup_{t\in\mathcal{T}_\rho}\frac{\chi_{\boldsymbol{b}}^2(t)}{w^2(t)} > u\right\}$$

$$(23) \qquad \geq \mathbb{P}\left\{\sup_{t\in\mathcal{T}}\frac{\chi_{\boldsymbol{b}}^2(t)}{w^2(t)} > u\right\}$$

$$\geq \sum_{i=1}^m p_i(u) - \sum_{1\leq i<j\leq m}\mathbb{P}\left\{\sup_{t\in[t_i-\rho,t_i+\rho]}\frac{\chi_{\boldsymbol{b}}^2(t)}{w^2(t)} > u, \sup_{t\in[t_j-\rho,t_j+\rho]}\frac{\chi_{\boldsymbol{b}}^2(t)}{w^2(t)} > u\right\},$$

where

$$p_i(u) = \mathbb{P}\left\{\sup_{t\in[t_i-\rho,t_i+\rho]}\frac{\chi_{\boldsymbol{b}}^2(t)}{w^2(t)} > u\right\}.$$

We first focus on the asymptotics of $p_i(u)$ as $u \to \infty$. Denote

$$Y(t) = \frac{w(t_1)}{w(t)}X(t), \quad t \in \mathcal{T}.$$

We have

$$p_i(u) = \mathbb{P}\left\{\sup_{t\in[t_i-\rho,t_i+\rho]}\sum_{l=1}^n b_l^2 Y_l^2(t) > w^2(t_1)u\right\}, \quad 1 \leq i \leq m,$$

where $\{Y_l\}_{i=1}^n$ is a sequence of independent copies of Gaussian process Y. It can be shown that, by **F1**, for any $i = 1, 2, \cdots, m$,

$$\sigma_Y(t) = \sqrt{\mathbb{E}\left((Y(t))^2\right)} = \frac{w(t_1)}{w(t)}, \quad t \in [t_i - \rho, t_i + \rho],$$

attains its maximum which is equal to 1 at the unique point t_i, and further

$$\sigma_Y(t_i + t) = 1 - \frac{a_i}{w(t_1)}|t|^{\beta_i}(1 + o(1)), \quad t \to 0.$$

Moreover, by (5)

$$1 - Corr\left(Y(t_i + t), Y(t_i + s)\right) = C(t_i)K^2(|t - s|)(1 + o(1)), \quad t \to 0.$$

Consequently, it follows from [8][Theorem 5.2] that, as $u \to \infty$,

$$(24) \qquad p_i(u) \sim \left(\prod_{l=k+1}^{n} (1 - b_l^2)^{-1/2} \right) \mathcal{M}_i(\beta_i, u) \; \Upsilon_k(w^2(t_1)u),$$

where $\Upsilon_k(\cdot)$ is given in (14) and

$$\mathcal{M}_i(\beta_i, u) = \begin{cases} 2a_i^{-1/\beta_i}(w(t_1))^{2/\alpha-1/\beta_i}(C(t_i))^{1/\alpha}\Gamma(1/\beta_i+1)\mathcal{H}_\alpha(q(u))^{-1}u^{-1/\beta_i}, & \text{for } \mathbf{C1}(\beta_i), \\ \mathcal{P}_\alpha^{a_i(w(t_1)C(t_i))^{-1}\mathcal{L}^\alpha} & \text{for } \mathbf{C2}(\beta_i), \\ 1, & \text{for } \mathbf{C3}(\beta_i). \end{cases}$$

In the sequel, we discuss the three scenarios $\mathbf{C1}(\beta), \mathbf{C2}(\beta), \mathbf{C3}(\beta)$ one-by-one.

$\underline{\mathbf{C1}(\beta)}$. Using the fact that $\beta = \max_{i=1}^{m} \beta_i$, we have that

$$\mathcal{M}_j(\beta_j, u) = o\left(\mathcal{M}_i(\beta_i, u) \right), \quad u \to \infty$$

for any $i \in V$ and $j \in V^c$. This implies that

$$(25) \qquad \sum_{i=1}^{m} p_i(u) \sim \sum_{i \in V} p_i(u) \sim \left(\prod_{l=k+1}^{n} (1 - b_l^2)^{-1/2} \right) \mathcal{M}(u) \; \Upsilon_k(w^2(t_1)u),$$

where $\mathcal{M}(\cdot)$ is given in (15). On the other hand, it follows directly from Lemma 4.1 that

$$\mathbb{P}\left\{ \sup_{t \in \mathcal{T}_\rho} \frac{\chi_{\boldsymbol{b}}^2(t)}{w^2(t)} > u \right\} \leq \exp\left(-\frac{\inf_{t \in \mathcal{T}_\rho} w^2(t)(\sqrt{u} - Q)^2}{2} \right)$$

holds for all $u > Q^2$, with Q some positive constant. Since further, by $\mathbf{F1}$,

$$\inf_{t \in \mathcal{T}_\rho} w^2(t) > w^2(t_1),$$

we have that

$$(26) \qquad \mathbb{P}\left\{ \sup_{t \in \mathcal{T}_\rho} \frac{\chi_{\boldsymbol{b}}^2(t)}{w^2(t)} > u \right\} = o\left(\mathcal{M}(u) \; \Upsilon_k(w^2(t_1)u) \right), \quad u \to \infty.$$

Moreover, since for any $i \neq j$

$$\mathbb{P}\left\{ \sup_{t \in [t_i-\rho, t_i+\rho]} \frac{\chi_{\boldsymbol{b}}^2(t)}{w^2(t)} > u, \sup_{t \in [t_j-\rho, t_j+\rho]} \frac{\chi_{\boldsymbol{b}}^2(t)}{w^2(t)} > u \right\}$$

$$\leq \mathbb{P}\left\{ \sup_{t \in [t_i-\rho, t_i+\rho]} \chi_{\boldsymbol{b}}^2(t) > w^2(t_1)u, \sup_{t \in [t_j-\rho, t_j+\rho]} \chi_{\boldsymbol{b}}^2(t) > w^2(t_1)u \right\}.$$

we have from Lemma 4.2 that, for all u large,

$$\mathbb{P}\left\{ \sup_{t \in [t_i-\rho, t_i+\rho]} \frac{\chi_{\boldsymbol{b}}^2(t)}{w^2(t)} > u, \sup_{t \in [t_j-\rho, t_j+\rho]} \frac{\chi_{\boldsymbol{b}}^2(t)}{w^2(t)} > u \right\} \leq \exp\left(-\frac{(2w(t_1)\sqrt{u} - Q_{i,j})^2}{2(2+2\eta)} \right), \quad 1 \leq i < j \leq m,$$

with $Q_{i,j}$'s some positive constants and $\eta \in (0, 1)$. Therefore, as $u \to \infty$,

$$(27) \qquad \sum_{1 \leq i < j \leq m} \mathbb{P}\left\{ \sup_{t \in [t_i-\rho, t_i+\rho]} \frac{\chi_{\boldsymbol{b}}^2(t)}{w^2(t)} > u, \sup_{t \in [t_j-\rho, t_j+\rho]} \frac{\chi_{\boldsymbol{b}}^2(t)}{w^2(t)} > u \right\} = o\left(\mathcal{M}(u) \; \Upsilon_k(w^2(t_1)u) \right).$$

Combining (25)–(27) with (23) we establish the claim of $\mathbf{C1}(\beta)$.

$\underline{\mathbf{C2}(\beta)}$. In this case, we have that (24) holds with

$$\mathcal{M}_i(\beta_i, u) = \begin{cases} \mathcal{P}_\alpha^{a_i(w(t_1)C(t_i))^{-1}\mathcal{L}^\alpha}, & i \in V \\ 1, & i \in V^c. \end{cases}$$

Consequently,

$$\sum_{i=1}^{m} p_i(u) \sim \left(\prod_{l=k+1}^{n} (1 - b_l^2)^{-1/2} \right) \left(\sum_{i \in V} \mathcal{P}_\alpha^{a_i(w(t_1)C(t_i))^{-1}\mathcal{L}^\alpha} + \sharp V^c \right) \Upsilon_k(w^2(t_1)u).$$

Note that (26) and (27) still hold. Similarly as the case $\mathbf{C1}(\beta)$, we establish the claim of $\mathbf{C2}(\beta)$.

$\underline{\mathbf{C3}(\beta)}$. In this case, we have that (24) holds with

$$\mathcal{M}_i(\beta_i, u) = 1, \quad 1 \le i \le m.$$

Consequently,

$$\sum_{i=1}^{m} p_i(u) \sim m \left(\prod_{l=k+1}^{n} (1 - b_l^2)^{-1/2} \right) \Upsilon_k(w^2(t_1)u).$$

Similarly as before, the claim of $\mathbf{C3}(\beta)$ follows.

(ii). By $\mathbf{F2}$ we have for any sufficiently small $\varepsilon > 0$ it holds that

$$\inf_{t \in \mathcal{T}_\varepsilon} w(t) > w(c_1), \quad \text{with } \mathcal{T}_\varepsilon = \mathcal{T} \setminus \bigcup_{i=1}^{m} [c_i - \varepsilon, d_i + \varepsilon].$$

Similarly to (23) we have

(28)
$$\sum_{i=1}^{m} \mathbb{P} \left\{ \sup_{t \in [c_i - \epsilon, d_i + \epsilon]} \frac{\chi_{\boldsymbol{b}}^2(t)}{w^2(t)} > u \right\} + \mathbb{P} \left\{ \sup_{t \in \mathcal{T}_\varepsilon} \frac{\chi_{\boldsymbol{b}}^2(t)}{w^2(t)} > u \right\}$$
$$\ge \mathbb{P} \left\{ \sup_{t \in \mathcal{T}} \frac{\chi_{\boldsymbol{b}}^2(t)}{w^2(t)} > u \right\}$$
$$\ge \sum_{i=1}^{m} \mathbb{P} \left\{ \sup_{t \in [c_i, d_i]} \frac{\chi_{\boldsymbol{b}}^2(t)}{w^2(t)} > u \right\} - \sum_{1 \le i < j \le m} \mathbb{P} \left\{ \sup_{t \in [c_i, d_i]} \frac{\chi_{\boldsymbol{b}}^2(t)}{w^2(t)} > u, \sup_{t \in [c_j, d_j]} \frac{\chi_{\boldsymbol{b}}^2(t)}{w^2(t)} > u \right\}.$$

Next, we have from $\mathbf{F2}$ that for $1 \le i \le m$

$$\mathbb{P} \left\{ \sup_{t \in [c_i, d_i]} \frac{\chi_{\boldsymbol{b}}^2(t)}{w^2(t)} > u \right\} = \mathbb{P} \left\{ \sup_{t \in [c_i, d_i]} \chi_{\boldsymbol{b}}^2(t) > w^2(c_1)u \right\},$$

$$\mathbb{P} \left\{ \sup_{t \in [c_i - \varepsilon, d_i + \varepsilon]} \frac{\chi_{\boldsymbol{b}}^2(t)}{w^2(t)} > u \right\} \le \mathbb{P} \left\{ \sup_{t \in [c_i - \varepsilon, d_i + \varepsilon]} \chi_{\boldsymbol{b}}^2(t) > w^2(c_1)u \right\}.$$

It is noted that the result in Theorem 2.1 of [32] also holds when $g(t) = 0$. Thus, it follows from that result, as $u \to \infty$,

$$\mathbb{P} \left\{ \sup_{t \in [c_i, d_i]} \chi_{\boldsymbol{b}}^2(t) > w^2(c_1)u \right\} \sim \prod_{j=k+1}^{n} (1 - b_j^2)^{-1/2} \mathcal{H}_\alpha \int_{c_i}^{d_i} (C(t))^{1/\alpha} dt \, (q(w^2(c_1)u))^{-1} \Upsilon_k(w^2(c_1)u),$$

$$\mathbb{P} \left\{ \sup_{t \in [c_i - \varepsilon, d_i + \varepsilon]} \chi_{\boldsymbol{b}}^2(t) > w^2(c_1)u \right\} \sim \prod_{j=k+1}^{n} (1 - b_j^2)^{-1/2} \mathcal{H}_\alpha \int_{c_i - \varepsilon}^{d_i + \varepsilon} (C(t))^{1/\alpha} dt \, (q(w^2(c_1)u))^{-1} \Upsilon_k(w^2(c_1)u).$$

Moreover, since

$$\mathbb{P} \left\{ \sup_{t \in [c_i, d_i]} \frac{\chi_{\boldsymbol{b}}^2(t)}{w^2(t)} > u, \sup_{t \in [c_j, d_j]} \frac{\chi_{\boldsymbol{b}}^2(t)}{w^2(t)} > u \right\}$$

$$= \mathbb{P} \left\{ \sup_{t \in [c_i, d_i]} \chi_{\boldsymbol{b}}^2(t) > w^2(c_1)u, \sup_{t \in [c_j, d_j]} \chi_{\boldsymbol{b}}^2(t) > w^2(c_1)u \right\},$$

we have from Lemma 4.2 that, for all u large,

$$\mathbb{P}\left\{\sup_{t\in[c_i,d_i]}\frac{\chi_{\boldsymbol{b}}^2(t)}{w^2(t)}>u,\ \sup_{t\in[c_j,d_j]}\frac{\chi_{\boldsymbol{b}}^2(t)}{w^2(t)}>u\right\}\leq\exp\left(-\frac{(2w(c_1)\sqrt{u}-Q_{i,j})^2}{2(2+2\eta)}\right),\quad 1\leq i<j\leq m,$$

with $Q_{i,j}$'s some positive constants and $\eta\in(0,1)$. This implies that

$$\sum_{1\leq i<j\leq m}\mathbb{P}\left\{\sup_{t\in[c_i,d_i]}\frac{\chi_{\boldsymbol{b}}^2(t)}{w^2(t)}>u,\ \sup_{t\in[c_j,d_j]}\frac{\chi_{\boldsymbol{b}}^2(t)}{w^2(t)}>u\right\}=o\left((q(w^2(c_1)u))^{-1}\Upsilon_k(w^2(c_1)u)\right),\quad u\to\infty.$$

Moreover, Lemma 4.1 gives that

$$\mathbb{P}\left\{\sup_{t\in\mathcal{T}_\varepsilon}\frac{\chi_{\boldsymbol{b}}^2(t)}{w^2(t)}>u\right\}=o\left((q(w^2(c_1)u))^{-1}\Upsilon_k(w^2(c_1)u)\right),\quad u\to\infty.$$

Consequently, by letting $\varepsilon\to0$ we conclude that the claim in (ii) is established. This completes the proof.
□

Proof of Corollary 3.4: We have from Example 3.2, for $\rho_1>0$ and $\rho_2\in\mathbb{R}$,

$$\sup_{t\in(0,1)}\frac{\chi_{\boldsymbol{b}}^2(t)}{w_{\rho_1,\rho_2}^2(t)}<\infty\ \ a.s.$$

Furthermore, for the generic locally stationary Gaussian process $X=\overline{B}$ we have

$$(29)\qquad C(t)=\frac{1}{2t(1-t)},\qquad K(h)=\sqrt{|h|},\quad h\in(0,1).$$

Next, in order to apply Theorem 3.3 we analyze the function $w_{\rho_1,\rho_2}(t)$.
For simplicity, we define

$$f(x)=2\rho_1 x+2\rho_2\ln x,\quad x(t)=\ln\ln\left(\frac{e^2}{t(1-t)}\right).$$

Apparently,

$$w_{\rho_1,\rho_2}^2(t)=f(x(t)),\quad t\in(0,1),\quad \{x(t):t\in(0,1)\}=[\ln\ln(4e^2),\infty).$$

Since

$$\frac{\partial f(x)}{\partial x}=2\rho_1+\frac{2\rho_2}{x},\quad x\in[\ln\ln(4e^2),\infty),$$

the following three different cases will be discussed separately:

$$a).\ \rho_2>-\rho_1\ln\ln(4e^2);\quad b).\ \rho_2=-\rho_1\ln\ln(4e^2);\quad c).\ \rho_2<-\rho_1\ln\ln(4e^2).$$

$\underline{a).\ \rho_2>-\rho_1\ln\ln(4e^2)}$: In this case, we have

$$f'(x)=\frac{\partial f(x)}{\partial x}>0,\quad x\in[\ln\ln(4e^2),\infty),$$

which means that $f(x)$ attains its minimum over $[\ln\ln(4e^2),\infty)$ at the unique point $x_0=\ln\ln(4e^2)$, and $f'(x_0)>0$. Since further

$$x'(t)=\frac{\partial x(t)}{\partial t}=\frac{t(1-t)}{\ln\left(\frac{e^2}{t(1-t)}\right)}\left(t^{-1}(1-t)^{-2}-t^{-2}(1-t)^{-1}\right),$$

we conclude that the minimizer of $f(x(t))$ over $(0,1)$ is unique and equal to $t_1=1/2$, and $x(t_1)=x_0$, $x'(t_1)=0$. Next, we look at the Taylor expansion of $f(x(t)),t\in(0,1)$ at t_1. To this end, we calculate the derivatives of it. Clearly,

$$(30)\qquad\frac{\partial f(x(t))}{\partial t}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial t},\quad \frac{\partial^2 f(x(t))}{\partial t^2}=\frac{\partial^2 f}{\partial x^2}\left(\frac{\partial x}{\partial t}\right)^2+\frac{\partial f}{\partial x}\frac{\partial^2 x}{\partial t^2},$$

Then it follows that

$$\frac{\partial f(x(t))}{\partial t}\bigg|_{t=t_1} = 0, \quad \frac{\partial^2 f(x(t))}{\partial t^2}\bigg|_{t=t_1} = \frac{\partial f}{\partial x}\bigg|_{x=x_0}\frac{\partial^2 x}{\partial t^2}\bigg|_{t=t_1} = \frac{16}{\ln(4e^2)}\left(\rho_1 + \frac{\rho_2}{\ln\ln(4e^2)}\right) =: Q_1 > 0.$$

Thus, by Taylor expansion

$$f(x(t)) - f(x(t_1)) = \frac{Q_1}{2}(t-t_1)^2(1+o(1)), \quad t \to t_1$$

which yields that

$$(31) \qquad w_{\rho_1,\rho_2}(t) = w_{\rho_1,\rho_2}(t_1) + \frac{Q_1}{4w_{\rho_1,\rho_2}(t_1)}(t-t_1)^2(1+o(1)), \quad t \to t_1.$$

Moreover, by (29) and (31)

$$C(t_1) = 2, \quad \alpha = 1 < 2 = \beta, \quad q(u) = u^{-1}.$$

Since further $\Gamma(1/2+1) = 1/2\sqrt{\pi}$ and $\mathcal{H}_1 = 1$, by applying Theorem 3.3 we conclude that the claim in (a) is established.

b). $\rho_2 = -\rho_1 \ln\ln(4e^2)$: In this case, we have that $f(x)$ attains its minimum over $[\ln\ln(4e^2), \infty)$ at the unique point $x_0 = \ln\ln(4e^2)$, but with $f'(x_0) = 0$. Further, the minimizer of $f(x(t))$ over $(0,1)$ is unique and equal to $t_1 = 1/2$, and $x(t_1) = x_0$, $x'(t_1) = 0$. Thus, we have from (30) that

$$\frac{\partial f(x(t))}{\partial t}\bigg|_{t=t_1} = 0, \quad \frac{\partial^2 f(x(t))}{\partial t^2}\bigg|_{t=t_1} = 0.$$

Next, let us calculate higher-order derivatives of $f(x(t))$. We have

$$\frac{\partial^3 f(x(t))}{\partial t^3} = \frac{\partial^3 f}{\partial x^3}\left(\frac{\partial x}{\partial t}\right)^3 + 3\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial t}\frac{\partial^2 x}{\partial t^2} + \frac{\partial f}{\partial x}\frac{\partial^3 x}{\partial t^3},$$

$$\frac{\partial^4 f(x(t))}{\partial t^4} = \frac{\partial^4 f}{\partial x^4}\left(\frac{\partial x}{\partial t}\right)^4 + 6\frac{\partial^3 f}{\partial x^3}\left(\frac{\partial x}{\partial t}\right)^2\frac{\partial^2 x}{\partial t^2} + 3\frac{\partial^2 f}{\partial x^2}\left(\frac{\partial^2 x}{\partial t^2}\right)^2 + 4\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial t}\frac{\partial^3 x}{\partial t^3} + \frac{\partial f}{\partial x}\frac{\partial^4 x}{\partial t^4}.$$

This implies that

$$\frac{\partial^3 f(x(t))}{\partial t^3}\bigg|_{t=t_1} = 0, \quad \frac{\partial^4 f(x(t))}{\partial t^4}\bigg|_{t=t_1} = 3\frac{\partial^2 f}{\partial x^2}\bigg|_{x=x_0}\left(\frac{\partial^2 x}{\partial t^2}\bigg|_{t=t_1}\right)^2 = \frac{384\rho_1}{\ln\ln(4e^2)(\ln(4e^2))^2} =: Q_2 > 0.$$

Therefore, by Taylor expansion we conclude that

$$(32) \qquad w_{\rho_1,\rho_2}(t) = w_{\rho_1,\rho_2}(t_1) + \frac{Q_2}{48w_{\rho_1,\rho_2}(t_1)}(t-t_1)^4(1+o(1)), \quad t \to t_1.$$

Similarly as in (a), the claim of (b) follows by applying Theorem 3.3.

c). $\rho_2 < -\rho_1 \ln\ln(4e^2)$: In this case, we have that $f(x)$ attains its minimum over $[\ln\ln(4e^2), \infty)$ at an inner point $x_0 = -\rho_2/\rho_1$, for which $f'(x_0) = 0$. Furthermore, the minimizer of $f(x(t))$ over $(0,1)$ are two distinct points $t_1 = \frac{1+\sqrt{1-4e^{2-e^{-\rho_2/\rho_1}}}}{2}$ and $t_2 = \frac{1-\sqrt{1-4e^{2-e^{-\rho_2/\rho_1}}}}{2}$, for which $x'(t_i) \neq 0, i = 1,2$. Thus, we have, for $i = 1,2$,

$$\frac{\partial f(x(t))}{\partial t}\bigg|_{t=t_i} = 0, \quad \frac{\partial^2 f(x(t))}{\partial t^2}\bigg|_{t=t_i} = \frac{\partial^2 f}{\partial x^2}\bigg|_{x=x_0}\left(\frac{\partial x}{\partial t}\bigg|_{t=t_i}\right)^2 = \frac{2\rho_1^2}{-\rho_2}Q_3 > 0,$$

where, by symmetry of $x(t), t \in (0,1)$,

$$Q_3 := \left(\frac{\partial x}{\partial t}\bigg|_{t=t_1}\right)^2 = \left(\frac{\partial x}{\partial t}\bigg|_{t=t_2}\right)^2 > 0.$$

Consequently, by Taylor expansion we conclude that

$$(33) \qquad w_{\rho_1,\rho_2}(t) = w_{\rho_1,\rho_2}(t_i) + \frac{\rho_1^2 Q_3}{-2\rho_2 w_{\rho_1,\rho_2}(t_i)}(t - t_i)^2(1 + o(1)), \quad t \to t_i.$$

In addition,

$$w_{\rho_1,\rho_2}(t_1) = w_{\rho_1,\rho_2}(t_2) = \sqrt{2\rho_2(\ln(-\rho_2) - \ln(\rho_1) - 1)}.$$

Similarly as in (a), the claim of (c) follows by applying Theorem 3.3. This completes the proof. □

Proof of Corollary 3.5: First note that

$$\lim_{h \to 0} \frac{1 - \mathbb{E}\left(\overline{B}_H(t), \overline{B}_H(t + h)\right)}{|h|^{2H}} = \frac{1}{2t^{2H}}$$

holds uniformly in $t \in I$, for any compact interval I in $(0, 1]$. This means that $\overline{B}_H$ is a locally stationary Gaussian process with

$$C(t) = \frac{1}{2t^{2H}}, \quad K(h) = |h|^H, \quad \alpha = 2H.$$

We shall first discuss the finiteness of $\sup_{t \in (0,1]} \frac{\chi_b^2(t)}{w_{\rho,\varepsilon}^2(t)}$, for which we only need to verify the conditions in Theorem 3.1 for the case where $S = 0$. First, condition $\mathbf{A}(0)$ is satisfied by $w_{\rho,\varepsilon}$, and clearly

$$f(0) = \frac{1}{2^{1/(2H)}} \int_{1/2}^0 t^{-1} dt = -\infty.$$

Further, we have from the calculations in the proof of Corollary 2.7 in [32] that $\mathbb{E}\left(\overline{B}_H(t), \overline{B}_H(s)\right) < 1$ for $s \neq t, s, t \in (0, 1]$, and conditions $\mathbf{B(0)}$ and $\mathbf{C(0)}$ are satisfied by $\overline{B}_H(t), t \in (0, 1]$. Moreover, we have that

$$J_{c,w_{\rho,\varepsilon}}(0) \leq \frac{1}{2^{1/(2H)}} \left(\frac{1}{(\ln(e^2/\epsilon))^{\rho c}} \int_{\varepsilon \wedge 1/2}^{1/2} \frac{1}{t} dt + \int_0^\varepsilon \frac{1}{t(\ln(e^2/t))^{\rho c}} dt \right) < \infty$$

for any $c > 1/\rho$. Consequently, it follows from Theorem 3.1 that

$$\sup_{t \in (0,1]} \frac{\chi_b^2(t)}{w_{\rho,\varepsilon}^2(t)} < \infty \quad a.s.$$

Next, since by definition $w_{\rho,\varepsilon}$ attains its minimum over $(0, 1]$ on the interval $[\varepsilon, 1]$, we conclude from (ii) of Theorem 3.3 that the claim follows. □

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