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Polyfunctions over Commutative Rings

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Abstract

A function $f: R \to R$, where R is a commutative ring with unit element, is called *polyfunction* if it admits a polynomial representative $p \in R[x]$. Based on this notion we introduce ring invariants which associate to R the numbers s(R) and s(R'; R), where R' is the subring generated by 1. For the ring R = $\mathbb{Z}/n\mathbb{Z}$ the invariant s(R) coincides with the number theoretic *Smarandache* or *Kempner function* s(n). If every function in a ring R is a polyfunction, then R is a finite field according to the Rédei-Szele theorem, and it holds that s(R) = |R|. However, the condition s(R) = |R| does not imply that every function $f: R \to R$ is a polyfunction. We classify all finite commutative rings R with unit element which satisfy s(R) = |R|. For infinite rings R, we obtain a bound on the cardinality of the subring R' and for s(R'; R) in terms of s(R). In particular we show that $|R'| \leq s(R)!$. We also give two new proofs for the Rédei-Szele theorem which are based on our results.

1 Introduction

For a commutative ring R with unit element, a function $f : R \to R$ is said to be a *polyfunction* if there exists a polynomial $p \in R[x]$ such that f(x) = p(x)for all $x \in R$ (see [11, 9], and also [1, 2] for a discussion on polyfunctions from $\mathbb{Z}_m \to \mathbb{Z}_n$). The set of polyfunctions over R equipped with pointwise addition and multiplication forms a subring

$$G(R) := \{ f : R \to R, \exists p \in R[x] \ \forall x \in R \implies p(x) = f(x) \}$$

of \mathbb{R}^R and will be called the *ring of polyfunctions* over \mathbb{R} . The polynomials in $\mathbb{R}[x]$ which represent the zero element in $G(\mathbb{R})$ are called *null-polynomials* (see [13]). If S is a subring of \mathbb{R} , then

$$G(S;R) := \{ f: R \to R, \exists p \in S[x] \ \forall x \in R \implies p(x) = f(x) \},\$$

is a natural subring of G(R). In particular, the subring R' generated by the unit element 1 in R gives rise to the *integer polyfunctions* G(R'; R). Instead of restricting the ring of allowed coefficients as in the construction for G(S; R), one obtains other rings of polyfunctions by restricting the domain: The ring

$$\{f: S \to R, \exists p \in R[x] \ \forall x \in S \implies p(x) = f(x)\}$$

e.g. contains G(R) as a subring.

If S is a subring of R, a characteristic number connected to S and R is the minimal degree m such that the function $x \mapsto x^m$ can be represented by a polynomial in S[x] of degree strictly smaller than m. Then, in particular, every function in G(S; R) has a polynomial representative of degree strictly less than m. We set

$$s(S; R) := \min\{m \in \mathbb{N}, \exists p \in S[x], \deg(p) < m, \forall x \in R \implies p(x) = x^m\}$$

and s(R) := s(R; R) for brevity. We set $s(S; R) := \infty$ if no function $x \mapsto x^m$ can be represented by a polynomial of degree strictly smaller than m.

Trivially, we have $s(S; R) \ge s(T; R) \ge s(R)$ whenever $S \subset T$ are subrings of R. On the other hand, we will see in Section 3, that $s(R'; R) < \infty$ is bounded in terms of s(R) if $s(R) < \infty$.

Clearly, if two rings R_1, R_2 are isomorphic, then $s(R_1) = s(R_2)$ and $s(R'_1, R_1) = s(R'_2, R_2)$. In other words, $R \mapsto s(R)$ and $R \mapsto s(R', R)$ are ring invariants.

The function s, which associates to a given ring R the number $s(R) \in \mathbb{N} \cup \{\infty\}$ has been introduced in [5] and is called *Smarandache function*. This naming stems from the fact, that for all $2 \leq n \in \mathbb{N}$, the map $n \mapsto s(\mathbb{Z}/n\mathbb{Z})$ coincides with the well-known number theoretic Smarandache or Kempner function s (see [5, Theorem 2]) defined by

$$s(n) := \min\{k \in \mathbb{N}, n \mid k!\}$$

$$(1)$$

(see Lucas [8], Neuberg [10] and Kempner [6]). In fact, Legendre has already studied aspects of the function s(n): In [7] he showed that if $n = p^{\mu}$ for some prime p and $1 \leq \mu \in \mathbb{N}$, then s(n) verifies

$$s(n) = \mu(p-1) + a_0 + a_1 + \ldots + a_k$$

where the numbers a_i are the digits of s(n) in base p. i.e. $s(n) = a_k p^k + \ldots + a_0$ and $0 \leq a_i < p$. We refer to Dickson [3, p. 263–265] for the history of the function s(n).

In a finite field F, every function is a polyfunction as a polynomial respresentative of a function $f: F \to F$ is, e.g., given by the Lagrange interpolation polynomial for f. This representation property characterizes finite fields among commutative rings with unit element (see [12]):

Theorem 1 (Rédei, Szele). If R is a commutative ring with unit element then R is a finite field if and only if every function $f : R \to R$ can be represented by a polynomial in R[x].

We will include two short alternative proofs of this theorem in Section 4. For finite fields F, one has s(F) = |F|, so in view of Theorem 1, it is natural to ask what can be said about commutative rings R with unit element for which s(R) = |R| holds true. Note that if R is a finite ring, it trivially holds that $s(R) \leq |R|$, as the polynomial

$$p(x) = \prod_{y \in R} (x - y)$$

is a normed null-polynomial of degree |R|.

The following theorem (which will be restated below for the reader's convenience as Theorem 3), answers the above question and classifies all finite commutative rings R with unit element that satisfy s(R) = |R|:

Theorem. Let R be a finite commutative ring with unit element. Then, s(R) = |R| holds if and only if R is one of the following:

- (a) R is a finite field, or
- (b) R is \mathbb{Z}_4 , or
- (c) R is the ring ρ with four elements $\{0, 1, a, 1+a\}$ with 1+1=0 and $a^2=0$.

Remarks:

- 1. The ring ρ is not a field since it has zero divisors, and since it is of characteristic 2, it is not isomorphic to \mathbb{Z}_4 .
- 2. Observe the similarity between this result and the fact that for $n \ge 2$, the usual Smarandache function satisfies s(n) = n if and only if n is prime or n = 4.

Section 2 is devoted to the proof of this theorem. In Section 3 we discuss infinite rings and show that for an infinite commutative ring R with unit element and $s(R) < \infty$, we obtain an upper bound for |R'| and for s(R'; R) in terms of s(R), where R' is the subring of R generated by 1. Finally, in Section 4, we give two proofs of Theorem 1 – a direct one and one that is based on Theorem 3.

Throughout the article, $n \ge 2$ will denote a natural number, and $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ is the ring of integers modulo n, and we write $a \mid b$ if b is an integer multiple of a.

2 Polyfunctions over Finite Rings

Theorem 1 answered the question, when a ring R has the property, that every function $f: R \to R$ can be represented by a polynomial in R[x]. For finite rings a necessary (but not sufficient) condition for this property to hold is

$$s(R) = |R|,\tag{2}$$

(see Theorem 3 below). In this section, we want to address the question for which finite rings, equation (2) holds. The first step to answer this, is the following proposition:

Proposition 2. If R is a commutative ring with unit element and with zero divisors then either

- (a) there exist $a, b \in R \setminus \{0\}$ with $a \neq b$ and ab = 0, or
- (b) R is \mathbb{Z}_4 , or
- (c) R is the ring ρ with four elements $\{0, 1, a, 1+a\}$ with 1+1=0 and $a^2=0$.

Proof

Let us assume that in R the implication holds: if $u, v \in R \setminus \{0\}$ and uv = 0 then it follows u = v. Let $a \in R \setminus \{0\}$ be a zero divisor: $a^2 = 0$. Thus, if x is an element

in R with ax = 0, we have either x = 0 or x = a. Notice that for all $u \in R$ we have

$$a(au) = 0$$

and hence for all $u \in R$

$$au = 0$$
 or $a(u - 1) = 0$.

Hence, we have only the four cases u = 0 or u = a or u = 1 or u = 1 + a. If 1 + 1 = a, then $R = \mathbb{Z}_4$, if 1 + 1 = 0, then R is the ring ρ in (c).

We can now prove the main result of this section:

Theorem 3. Let R be a finite commutative ring with unit element. Then, s(R) = |R| holds if and only if R is one of the following:

- (a) R is a finite field, or
- (b) R is \mathbb{Z}_4 , or
- (c) R is the ring ρ with four elements $\{0, 1, a, 1+a\}$ with 1+1=0 and $a^2=0$.

Proof

If R is not a field and not \mathbb{Z}_4 and not the ring ρ , then, according to Proposition 2, R is a ring with $a, b \in \mathbb{R} \setminus \{0\}$ such that ab = 0 and with $a \neq b$. Then

$$(x-a)(x-b)\prod_{z\in R\setminus\{a,b,0\}}(x-z)$$

is a normed null-polynomial of degree |R| - 1. Therefore s(R) < |R|.

To prove the opposite direction, we go through the three cases:

(a) If R is a field, then a polynomial of degree n has at most n roots. Hence, s(R) = |R|.

(b) If R is \mathbb{Z}_4 , then (by [5, Theorem 2]) $s(\mathbb{Z}_4) = s(4) = 4 = |\mathbb{Z}_4|$.

(c) If R is the ring ρ with elements $\{0, 1, a, 1+a\}$ and with 1+1=0 and $a^2=0$, we have to prove that s(R) = 4. Assume by contradiction, that $p(x) \in R[x]$ is a normed null-polynomial of degree 3. Since p(0) = p(1) = 0, p(x) must be of the form

$$p(x) = x(x+1)(\xi+x).$$

From p(a) = 0, it follows that $a\xi = 0$ and from p(a+1) = 0 it subsequently follows that a = 0 which is a contradiction.

3 Infinite Rings

In this section R is a commutative ring with unit element and R' the subring of R which is generated by 1. We will need the following lemma, which is a corollary of [5, Lemma 4, p.4]:

Lemma 4. For all $k, n \in \mathbb{N} \cup \{0\}, k \leq n$ we have

$$\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} j^k = \delta_{kn} n!$$

(with the convention $0^0 := 1$).

Proposition 5. If $s(R) < \infty$ then R' is a finite ring and |R'||s(R)!.

Remark: We notice, that $s(R) < \infty$ may hold even if R is an infinite ring. As an example consider the ring

$$R = \mathbb{Z}_2[x_1, x_2, \ldots] / \{x_1^2, x_2^2, \ldots\}$$

in which all $u \in R$ satisfy the relation $u^4 = u^2$. On the other hand, if R is finite, we trivially have $s(R) \leq |R|$.

Proof of Proposition 5

By assumption, for n = s(R) there exist coefficients $a_i \in R$, $i \in \{0, 1, ..., n-1\}$, such that for all $u \in R$ we have

$$u^n - \sum_{i=0}^{n-1} a_i u^i = 0.$$
(3)

We denote

$$\underbrace{1+1+\ldots+1}_{m \text{ times}} \in R'$$

by \bar{m} . Then, by Lemma 4, we have for $k \leq n$

$$\sum_{j=0}^{n} \overline{(-1)^{n-j} \binom{n}{j} j^k} = \overline{\delta_{kn} n!}$$

$$\tag{4}$$

Hence, it follows from (3) that

$$0 = \sum_{j=0}^{n} \overline{(-1)^{n-j} \binom{n}{j}} \left(\overline{j}^n - \sum_{i=0}^{n-1} a_i \overline{j}^i \right) = \\ = \sum_{j=0}^{n} \overline{(-1)^{n-j} \binom{n}{j} j^n} - \sum_{i=0}^{n-1} a_i \sum_{j=0}^{n} \overline{(-1)^{n-j} \binom{n}{j} j^i} = \overline{n!}$$

where the last equality follows from (4).

Remark: As the example $R = \mathbb{Z}_{n!}$ shows, the estimate on the size of R' emerging from Proposition 5, $|R'| \leq s(R)!$, cannot be improved in general.

Lemma 6. If $n := s(R) < \infty$ then there exists a bound $\Lambda = n!^{(2n)^n n}$ for the cardinality of the orbits of the elements of R, i.e., for all $u \in R$ there holds

$$|\{u^k, k \in \mathbb{N}\}| \leqslant \Lambda.$$

Proof

As in the previous proof, we adopt (3). For $k \in \mathbb{N}$ let

$$M_k := \left\{ \prod_{i=0}^{n-1} a_i^{\varepsilon_i}, \varepsilon_i \in \{0, 1, \dots, k\} \right\}$$
$$N_k := \left\{ \sum_{\mu \in M_k} \overline{r_\mu} \, \mu, r_\mu \in \{0, 1, \dots, n! - 1\} \right\}$$

Observe that $|M_k| \leq (k+1)^n$ and $|N_k| \leq n!^{|M_k|}$. By Proposition 5 it follows that for $a, b \in N_k$, the sum a+b also belongs to N_k . On the other hand, by applying (3) to $u = a_j^2$, $j \in \{0, 1, \ldots, n-1\}$, we obtain

$$a_j^{2n} = \sum_{i=0}^{n-1} a_i a_j^{2i},$$

and hence, $N_k = N_{k-1}$ for $k \ge 2n$. It follows for all $u \in R$ and all $k \in \mathbb{N}$ that u^k is of the form

$$u^k = \sum_{i=0}^{n-1} \mu_i(k) u^j$$

for certain coefficients $\mu_i(k) \in N_{2n-1}$ and hence $|\{u^k, k \in \mathbb{N}\}| \leq |N_{2n-1}|^n \leq \Lambda$.

Theorem 7. If $n := s(R) < \infty$ then $s(R'; R) \leq \operatorname{lcm}(\Lambda) + \Lambda$, where $\Lambda = n!^{(2n)^n n}$.

Remarks:

(a) Here lcm(n) denotes the least common multiple of the numbers in the set $\{1, 2, ..., n\}$.

(b) Since R' is contained in every subring T (with 1) of R, the given bound also holds for s(T; R).

Proof of Theorem 7

By Lemma 6, there exist for arbitrary $u \in R$ integers $l < k \leq \Lambda + 1$ such that $u^k = u^l$. Thus, we have

$$u^{\operatorname{lcm}(\Lambda)+\Lambda} = u^{\operatorname{lcm}(\Lambda)+\Lambda - \frac{\operatorname{lcm}(\Lambda)}{k-l}(k-l)} = u^{\Lambda}.$$

We conclude this section by an example of a ring R which has the property, that s(R) < s(R', R).

Example: Let $R = \mathbb{Z}_2[x]/\{x^3 + x^4\}.$

The following lemma shows that for this particular ring $s(R) \leq 4$.

Lemma 8. For all polynomials $P \in \mathbb{Z}_2[x]$ we have that

$$xP + (1+x)P^2 + P^4 \equiv 0 \mod (x^3 + x^4).$$

\mathbf{Proof}

We first consider the special case $P(x) = x^m$. We have to show, that

$$xx^m + (1+x)x^{2m} + x^{4m} = x^{m+1} + x^{2m} + x^{2m+1} + x^{4m} \equiv 0 \mod (x^3 + x^4).$$

This is readily checked:

$$m = 0: \qquad x + 1 + x + 1 \equiv 0 \mod (x^3 + x^4)$$

$$m = 1: \qquad x^2 + x^2 + x^3 + x^4 \equiv 0 \mod (x^3 + x^4)$$

$$m \ge 2: \qquad x^3 + x^3 + x^3 \pm x^3 \equiv 0 \mod (x^3 + x^4)$$

Now, for arbitrary P, the claim follows by additivity in $\mathbb{Z}_2[x]$:

$$x(P_1 + P_2) + (1+x)(P_1 + P_2)^2 + (P_1 + P_2)^4 = \sum_{i=1}^2 xP_i + (1+x)P_i^2 + P_i^4.$$

Remark: We leave it to the reader to verify, that in fact s(R) = 4.

Now, we show that $s(R'; R) \ge 6$.

Lemma 9. Let $a_i \in \mathbb{Z}_2$ be such that $\sum_{i=0}^5 a_k u^k = 0$ in R for all $u \in R$. Then $a_0 = \cdots = a_5 = 0$.

Proof

First, by choosing u to be the class of x in R (which we denote by \bar{x}), we obtain

$$a_0 + a_1 \bar{x} + a_2 \bar{x}^2 + (a_3 + a_4 + a_5) \bar{x}^3 = 0$$
 in R

and hence, we conclude that $a_0 = a_1 = a_2 = 0$ and $a_3 + a_4 + a_5 = 0$. Next, we choose u to be the class of 1 + x in R. Observing that

$$(1 + \bar{x})^3 = 1 + \bar{x} + \bar{x}^2 + \bar{x}^3 \text{ in } R$$
$$(1 + \bar{x})^4 = 1 + \bar{x}^4 = 1 + \bar{x}^3 \text{ in } R$$
$$(1 + \bar{x})^5 = 1 + \bar{x} \qquad \text{ in } R$$

we have

$$0 = a_3 u^3 + a_4 u^4 + a_5 u^5 =$$

= $(a_3 + a_4 + a_5) + (a_3 + a_5)\bar{x} + a_3\bar{x}^2 + (a_3 + a_4)\bar{x}^3$ in R

which immediately implies that $a_3 = a_4 = a_5 = 0$. This completes the proof.

Finally we prove that s(R'; R) = 6.

Lemma 10. For all $u \in R$ it holds that $u^3 + u^4 + u^5 + u^6 = 0$ in R.

Proof

Let u be the class of a polynomial $P \in \mathbb{Z}_2[x]$ in R.

First case: P(0) = 0. In this case, we have

$$P(x) = xQ(x)$$

$$P^{2}(x) \equiv x^{2}Q^{2}(x) \mod (x^{3} + x^{4})$$

$$P^{3}(x) \equiv x^{3}Q^{3}(x) \equiv x^{3}Q(1) \mod (x^{3} + x^{4})$$

$$P^{4}(x) \equiv x^{4}Q^{4}(x) \equiv x^{3}Q(1) \mod (x^{3} + x^{4})$$

and hence $P^3(x) \equiv P^4(x) \mod (x^3 + x^4)$. This proves the claim in this case.

Second case: P(0) = 1. In this case, we have

$$\begin{split} P(x) &= 1 + xQ(x) \\ P^2(x) &\equiv 1 + x^2Q^2(x) \mod (x^3 + x^4) \\ P^3(x) &\equiv (1 + xQ(x))(1 + x^2Q^2(x)) \equiv \\ &\equiv 1 + xQ(x) + x^2Q^2(x) + x^3Q(1) \mod (x^3 + x^4) \\ P^4(x) &\equiv 1 + x^4Q^4(x) \equiv 1 + x^3Q(1) \mod (x^3 + x^4) \\ P^5(x) &\equiv (1 + xQ(x))(1 + x^3Q(1)) \equiv 1 + xQ(x) \equiv P(x) \mod (x^3 + x^4) \end{split}$$

which allows to verify the claim easily.

4 Two Alternative Proofs of the Rédei-Szele Theorem

We start with a short direct proof of Theorem 1. Let R be a commutative ring with unit element. One implication is immediate:

Assume that R is a finite field and $f: R \to R$. Then the Lagrange interpolation polynomial

$$p(x) = \sum_{y \in R} f(y) p_y(x),$$

where

$$p_y(x) = \prod_{z \in R \setminus \{y\}} (x - z) \left(\prod_{z \in R \setminus \{y\}} (y - z)\right)^{-1},$$

represents f.

For the opposite implication, we assume that every function $f : R \to R$ can be represented by a polynomial in R[x]. In particular, for the function

$$f(x) := \begin{cases} -1, & \text{if } x = 0\\ 0, & \text{if } x \neq 0 \end{cases}$$

there exists a representing polynomial

$$\sum_{k=0}^{n} a_k x^k = f(x) \quad \text{ for all } x \in R.$$

Since $a_0 = f(0) = -1$, it follows that

$$x \underbrace{\sum_{k=1}^{n} a_k x^{k-1}}_{=x^{-1}} = \sum_{k=1}^{n} a_k x^k = 1 \quad \text{for all } x \in R \setminus \{0\}.$$

Hence, R is a field. Moreover, for all $x \in R$

$$0 = xf(x) = \sum_{k=0}^{n} a_k x^{k+1}.$$
(5)

The right hand side of (5) is a polynomial of degree n + 1 which (in the field R) has at most n + 1 roots. Hence, $|R| \leq n + 1$.

A second alternative proof uses the characterization of the rings for which s(R) = |R| (see Theorem 3). This condition is necessary for the property, that all functions from R to R have a polynomial representative. In order to rule out the case $R = \mathbb{Z}_4$, we use the following formula from [4, Theorem 6, p.9]: If p is a prime number and $m \in \mathbb{N}$, the number of polyfunctions over \mathbb{Z}_{p^m} is given by

$$\Psi(p^m) := |G(\mathbb{Z}_{p^m})| = \exp_p\left(\sum_{k=1}^m s(p^k)\right).$$

Here s denotes the usual number theoretic Smarandache function (see equation (1)), and $\exp_p(q) := p^q$ for better readability. It follows that there are $\Psi(4) = \Psi(2^2) = 2^{2+4} = 64$ polyfunctions over \mathbb{Z}_4 , but the number of functions from \mathbb{Z}_4 to \mathbb{Z}_4 equals $4^4 = 256$. The case $R = \rho$ is ruled out by explicit verification that

$$f(x) = \begin{cases} 0 & \text{for } x \neq 0 \text{ and} \\ 1 & \text{for } x = 0 \end{cases}$$

is not a polyfunction over ρ : Since $s(\rho) = 4$, it is enough to show that no polynomial $p \in \rho[x]$ of degree ≤ 3 represents f. Suppose there is

$$p(x) = \sum_{k=0}^{3} a_k x^k$$

representing f. Then $p(0) = a_0 = 1$ and $p(a) = 1 + a_1 a = 0$, which implies that $a_1 a = 1$ which is impossible since a does not have a multiplicative inverse. \Box

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